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Preliminary Efforts to Find Statistical Distributions Representing Total Vertical Error and Assigned Altitude Deviation

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16. Abstract

Preliminary explorations are underway to determine theoretical probability models which provide appropriate representation of aggregate total vertical error (TVE) and assigned altitude deviation (AAD). A family of candidate distributions has been identified for matching to sample distributions of TVE and AAD using the Method of Maximum Likelihood. Results to date are mixed, due in part to limited amounts of TVE data currently available.

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INTRODUCTION

PURPOSE.

The purpose of this paper is to describe a family of functional forms that analysts with the United States (U.S.) vertical separation program believe to have the expected characteristics of either total vertical error (TVE) or assigned altitude deviation (AAD), and to present some preliminary findings. This family of forms serves as the basis for current U.S. investigations into the appropriate statistical characterization of aggregate TVE and AAD.

BACKGROUND.

Whenever a large amount of data is collected, there is a need to summarize it in a manner that provides insight into its overall nature. If the data represent errors of a system under study, such as TVE or AAD, one useful summary is provided by grouping these data into intervals dependent on the size of the error magnitude and then counting the number of errors within a given interval. The data are then usually plotted in bar graphs with the width of the bar representing intervals of error magnitudes and the height of each bar representing the number of errors within a given interval. The resulting figure, a histogram, provides a visual summary of the collected data and serves as one starting point for further investigation. Typically, TVE and AAD histogram bars are higher for smaller errors and decrease in height as error magnitude increases; that is, samples of such errors typically consist of a large number of small errors and a small number of large errors. Given a belief that the errors follow a predetermined functional form, attempts are made to describe relevant parameters by matching functional forms to the histograms of errors. When a reasonable match is found, further studies usually refer to the functional form as representative of the error distribution.

FAMILY OF DISTRIBUTIONS CONSIDERED

The family of distributions considered appropriate for fitting aggregate TVE or AAD observations has been chosen in light of a priori belief about the overall characteristics of these errors. Specifically, distributions of TVE and AAD are assumed to be symmetric, unimodal, and of strictly decreasing frequency for increasing error magnitude. Further, it is anticipated that the larger errors might have a functional form different from that for smaller errors, with both forms satisfying all the stated assumptions.

The rationale behind the three assumptions stated in the previous paragraph is that both TVE and AAD are properties of aircraft systems actively attempting to operate with as little error as possible. Furthermore, aircrews and altimetry manufacturers indicate that the assumptions are quite reasonable as long as the aircraft system is attempting to maintain level flight at an assigned altitude and on a constant heading. In addition to these expert opinions, there exist several bodies of AAD data whose statistical analysis yielded results supporting the assumptions.

The need for a combination of functional forms to properly account for the combined frequencies of larger and smaller errors was anticipated because of a previous attempt at representing the aggregate AAD of aircraft in level flight at and above 29,000 feet over the Continental United States (reference 1). This analysis indicated that mixing functional forms into a single model by appropriate weighting served to describe AAD adequately. Since AAD is generally thought of as a subset of TVE, it is anticipated that TVE, too, can be described by such a mixture. However, until recently, little empirical evidence concerning TVE has been available.

Based on previous experience as reported in reference 1, analysts in the U.S. vertical separation studies have constructed a family of eleven distributions from which, it is hoped, will emerge functional forms appropriate for modelling distributions of either TVE or AAD. Members of this family will now be described, proceeding from the simplest to the most complex.

The first two distributions are the normal and double exponential. Their probability density functions are given under headings A and B of the appendix. These are among the most commonly employed symmetric and unimodal distributions and are usually described by two parameters, one for central location and one for spread. Since the mean, or location parameter, is assumed to be zero in present studies, these distributions are described by only one parameter. For the normal distribution (model A in the appendix), the spread parameter is the standard deviation, σ. For the double exponential distribution (model B in the appendix), the spread parameter is β, the standard deviation divided by the square root of 2. These two distributions are included for consideration in the hope that the aggregate of smaller TVE or AAD errors (usually referred to as the core of the data) can be adequately modelled by one or the other.

The third distribution considered is a mixture of two double exponentials (model C in the appendix). The intent is to model both the core and tails (larger errors) of the data by one continuous function. Assuming a zero for the mean value, this function can be described by three parameters: (1) a spread parameter for the double exponential representing the core of the data (β_1), (2) a spread parameter for the double exponential characterizing the tails (β_2), and (3) a mixing parameter (α) to weight the distributions' relative contributions. The spread parameter for the tail, β_2, must be larger than β_1, otherwise, there would be no need for a mixture and one double exponential would be adequate. The mixing parameter, α, must be between 0.0 and 1.0 so that the resulting function is a probability density. When α is close to 0.0, most of the errors in the core of the data are characterized by that portion of the function with spread parameter β_1 and most of the errors in the tail by that portion of the function with spread parameter β_2. As α approaches 1.0, β_1 approaches β_2 and there is a heavy mixing of both functions in the core of the data. This function was first applied to characterize aircraft lateral navigation performance (reference 2) and has subsequently been used in fitting the distribution of AAD. The fourth distribution considered is a similar combination of two normal distributions (model D in the appendix), and is also described by two spread parameters (σ_1, σ_2) and a mixing parameter (α).

The fifth distribution is a power exponential (model E in appendix). It is a generalization of the first two distributions. Assuming a mean of 0 feet, two parameters are required to describe this distribution: (1) a shape parameter (β) and (2) a spread parameter ($\emptyset$). The shape parameter can vary from 1.0 to a value approaching -1.0. With $\beta = 1.0$, the power exponential is equivalent to a double exponential; $\beta = 0$ produces a normal distribution. As β approaches -1.0, the power exponential approaches a uniform distribution. In summary, for $\beta > 0.0$, the distribution is leptokurtic (more peaked than normal) and for $\beta < 0.0$, the distribution is platykurtic (flatter than normal). This distribution was chosen to model a sample distribution of errors which appears to be either between a uniform and a normal or between a normal and a double exponential.

The sixth and seventh distributions are mixtures of a power exponential and a double exponential distribution (model F in the appendix). Assuming a mean of 0 feet, these distributions can be described by four parameters. The first two parameters are the shape (β_1) and spread parameter ($\emptyset$) for the power exponential, the third is a spread parameter for the double exponential (β_2), and the fourth is a mixing parameter (α). This distribution first found application in modelling Mode C altitude deviations where the core of the data was slightly less peaked than a double exponential (see source for model F in appendix). In that case, α was close to zero. If the mixing parameter were set close to 1.0, this distribution would switch from one that modelled the core of the data as a power exponential and the tail as a double exponential to one that models the core of the data as a double exponential and the tail as a power exponential. Hence, it is considered as two models, numbers six and seven.

The eighth and ninth distributions are similar mixtures of a power exponential and a normal distribution. Assuming a mean of 0 feet, these distributions can also be described by four parameters (model G in appendix), two describing the power exponential, one representing the normal, and the fourth indicating the mix of the two forms. As with the sixth and seventh models, if the mix is switched, the role which each function plays in describing the core or tail of the data will switch.

The tenth distribution is a mixture of two power exponentials with the same shape parameter. Assuming a mean of 0 feet, it also has four parameters: (1) a spread parameter for the core of the data, $\emptyset$, (2) a spread parameter for the tail of the data, $\emptyset_2$, (3) a shape parameter, β, and (4) a mixing parameter, α (model H in appendix). This distribution was suggested after further examination of aircraft lateral position-keeping errors (reference 3).

The eleventh distribution is the most complex of the family. It is the mixture of two power exponential distributions. Again, assuming a mean of 0 feet, it has five parameters. Two describe the core portion of the data ($\emptyset_1$, β_1), two describe the tail portion of the data ($\emptyset_2$, β_2), and one (α) specifies the mix (model I in appendix).

METHOD OF FIT

Criteria need to be developed in order to estimate those parameters of a given statistical distribution that provide the best match (fit) between an assumed model and a set of empirical data. For probability density functions, the procedure that has often been used is the Method of Maximum Likelihood developed by R.A. Fisher. This procedure has two major advantages and one major disadvantage in comparison to other possible techniques that might be applied to TVE or AAD data.

The first advantage is that the Method of Maximum Likelihood provides probability statements about parameter estimates. In addition, if the two-dimensional contours of equal probability are ellipsoidal, then it can be assumed that these parameter estimates are normally distributed and estimates of their covariances may be constructed. These covariances might, in turn, be used to construct model-estimated probability confidence levels of exceeding given values of the random variable whose emporical distribution has been fitted. That is, it is possible to construct confidence intervals for one minus the cumulative curve. Finally, in the aircraft spacing application, these intervals can be translated into overlap probability confidence bands for two aircraft at a planned vertical separation standard (reference 4).

A second major advantage is that it is possible to compare different models within the family to determine if a more complex model significantly improves fit of the empirical data. This test involves calculation of the likelihood ratio for two models (reference 5).

The major disadvantage of this procedure is that it appears to be less sensitive to the data structure within the tail portion of the data than it is to the structure of the core. This is disconcerting since the tail portion is likely to influence computations of overlap probabilities.

DATA SOURCE

Some TVE and AAD estimates resulting from the U.S. vertical separation program data collection effort are now available. Due to the data collection goal of obtaining a broad sample of U.S high-altitude operations, these data may reflect different proportions of various aircraft types than are usually observed in the airspace where data were collected. Notwithstanding, two groups of data have been assembled. The first consists of aircraft tracked in airspace over the U.S. eastern seaboard during both the summer and winter seasons. The direction of flight was generally north or south and the terrain was relatively flat. The second group consists of aircraft that were tracked over the U.S. Rocky Mountains during winter. The direction of these flights was generally east or west.

The amount of AAD data for each of these groups is much greater than the corresponding amount of TVE data. This is because AAD data were collected for a longer time period on each aircraft and also because the data collection and reduction process for AAD is much less complex than the corresponding process for TVE.

EXPLORATORY MODELLING EFFORT

This section of the paper will discuss exploratory efforts made in an attempt to model four sets of data--AAD for United States east coast and mountainous regions and the corresponding limited subsamples of TVE for the two regions.

Figures 1 and 2 indicate that neither aggregate of TVE data could be described by a single functional form within the distribution family. Using the scale of figure 1, for example, a single functional form would decrease (1) linearly relative to increasing error magnitude if it was a double exponential, (2) quadratically if it was a normal, (3) from linearly to quadratically if it was a power exponential with $\beta>0$, and (4) more rapidly than a quadratic if $\beta<0$.

Error frequencies in figure 1 decrease in an apparent quadratic manner in the core of the data. As error magnitude increases, the decrease appears to stop before once again exhibiting a somewhat quadratic shape. Figure 2 shows evidence of a quadratic decrease in the core. Then there is a more precipitous decrease, followed by a quadratic decrease before a final fall.

The initial model choice for these two data groups was a mixture of two power exponentials with the same β parameter (model H in appendix). Efforts to fit the model resulted in what appears to be a reasonable representation of the core. Fitting the model to each data set resulted in β values near zero, indicating the core shape following a normal distribution (with standard deviations of about 80 and 130 feet for east coast and mountainous areas, respectively). Interestingly, the calculated TVE standard deviation of the east coast data was 113 feet. It was significantly larger than that of its modelled core (80 feet), while the calculated and model-produced standard deviations of the mountainous data group were about the same at 133 feet and 130 feet, respectively. The overall modelled standard deviation for each data group was about the same as that calculated from the data for each group.

These models did not fit the tail portion of the data well. Because of the sparse data in this area and the possibility of regrouping these data in a different manner, no further efforts were made to refine the models.

Similarly, it was also evident from figures 3 and 4 that AAD data could not be described by a single functional form. However, both of these figures do show a decrease that appears to be somewhat linear for the core of the data, in contrast to the core of the TVE data which appears quadratic. The initial model choice for these data was a mixture distribution with a double exponential for the core and a power exponential for the tail (model F in appendix). This initial choice resulted in what appears to be a reasonable representation for the core of the data, with standard deviations of about 50 feet for both east coast and mountainous areas. In contrast, the tail portion of the AAD data for the east coast group appears to have a very different shape from that of the mountainous area. More specifically, it appears that the tail portion of the AAD data for the east coast set has a platykurtic shape. That is, the shape appears less peaked than that of a normal

distribution (which would be quadratic when plotted on semi-logarithmic scale as in figure 3). On the other hand, frequencies in the tail portion of the AAD data from mountainous area operations decrease somewhat more linearly.

CONCLUSIONS

Preliminary results indicate that total vertical error appear to have a core shape (small error pattern) that follows a power exponential distribution which is almost Gaussian, with standard deviations of about 80 feet for east coast data and about 130 feet for mountainous areas. By contrast, assigned altitude deviations appear to have a core shape that follows a double exponential distribution with standard deviations of about 50 feet for both east coast and mountainous regions.

Although these initial explorations have provided some insight, they have not resulted in an an adequate description of the tail portion (large error pattern) of either TVE or AAD data. Efforts to obtain a more satisfactory description are underway.

REFERENCES

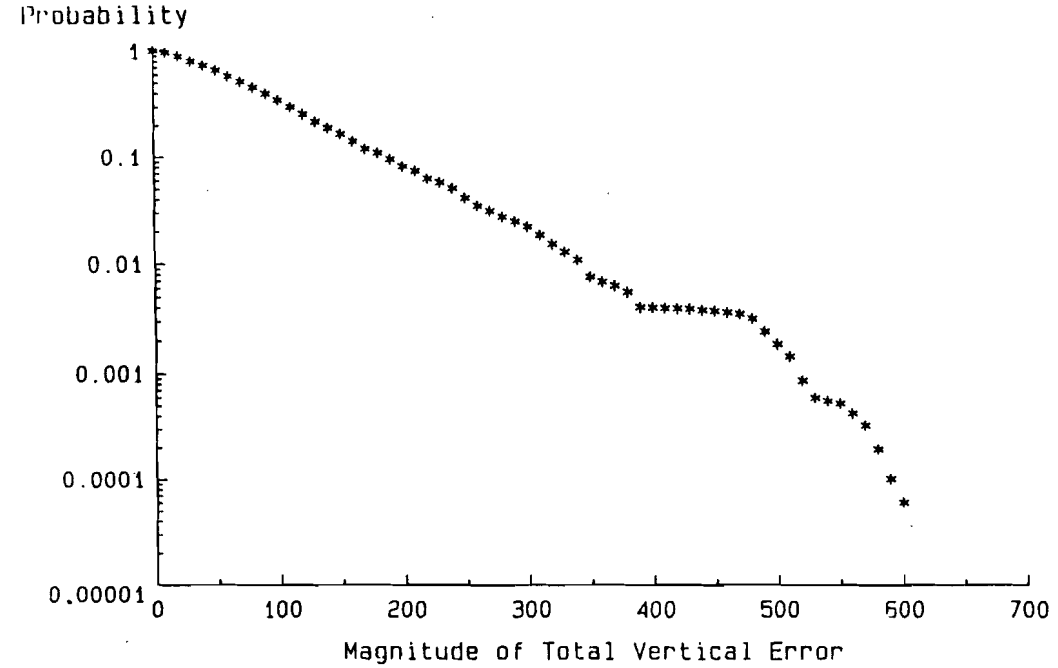
1. Rigolizzo, R., Reduction and Analysis of Mode C Altitude Data Collected at High Altitude Over the Continental United States, U.S. Department of Transportation, DOT/FAA/EM-82/9A, March 1982.

2. Busch, A., Colamosca, B., Navigational System Requirements via Collision Risk Model, International Institute of Navigation Conference, p. 4, May 1978.

3. Dsu, D.A. , Further Analysis of Position Errors in Navigation, Journal of Navigation, Volume 3, No. 3, p. 472, September 1980.

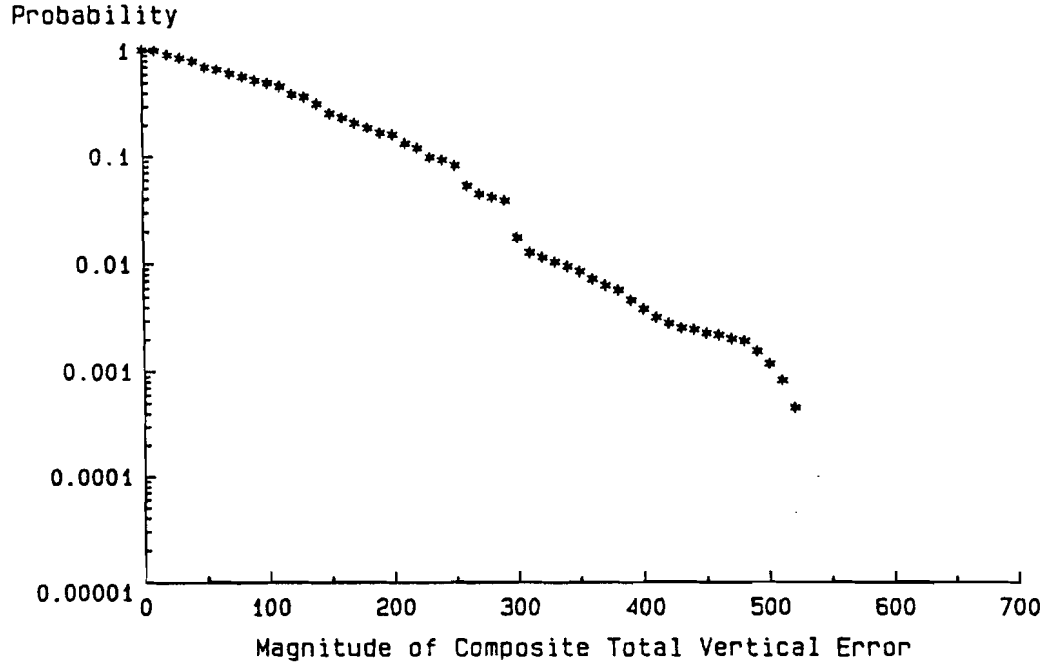
4. Ibid pp. 460-466.

5. Ibid pp. 458-459.



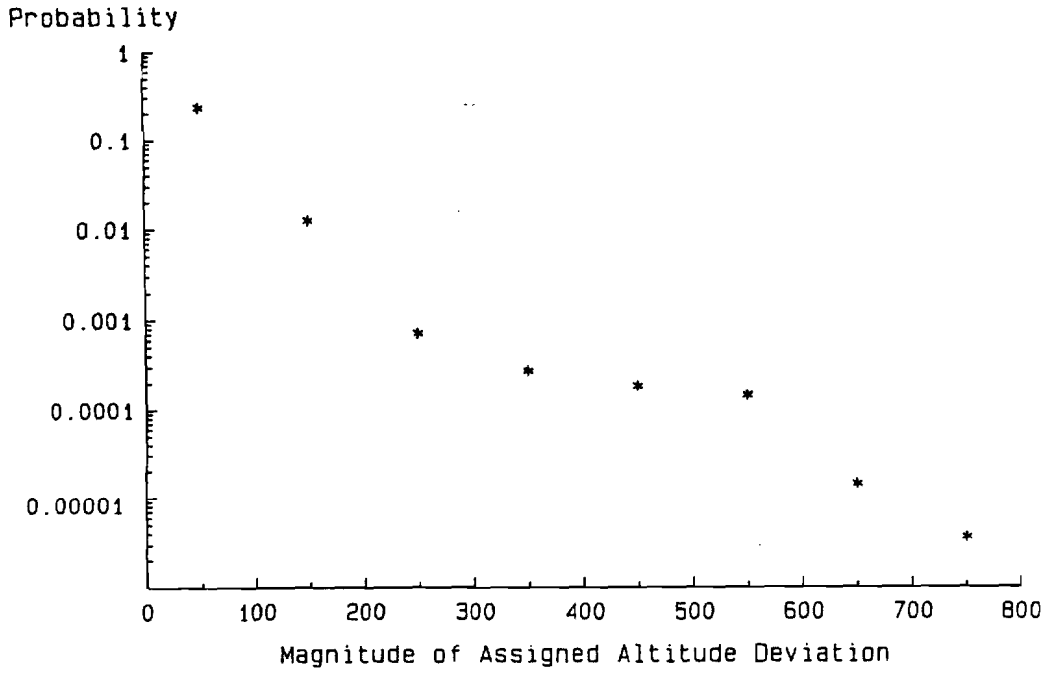
Total Aircraft = 318
Estimated Mean = -0.17
Estimated Std. Dev. = 113.21

FIGURE 1. ONE MINUS CUMULATIVE DISTRIBUTION OF TOTAL VERTICAL ERROR (WALLOPS 1983 SUBSAMPLE)



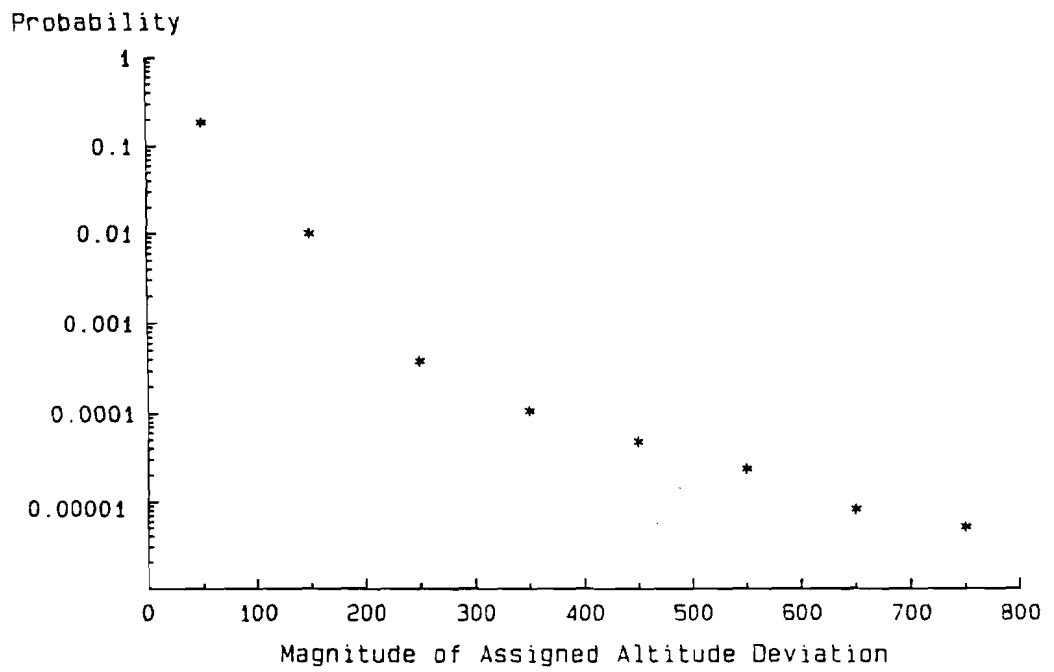
Total Aircraft = 63
Estimated Mean = -6.78
Estimated Std. Dev. = 133.33

FIGURE 2. ONE MINUS CUMULATIVE DISTRIBUTION OF COMPOSITE TOTAL VERTICAL ERROR (LONGMONT 1984 SUBSAMPLE)



Total Aircraft = 1844
Estimated Mean = 2.90
Estimated Std. Dev. = 52.72

FIGURE 3. ONE MINUS CUMULATIVE DISTRIBUTION OF ASSIGNED ALTITUDE DEVIATION (WALLOPS 1983, 1984, and 1985 SAMPLES)



Total Aircraft = 869
Estimated Mean = -1.94
Estimated Std. Dev. = 46.84

FIGURE 4. ONE MINUS CUMULATIVE DISTRIBUTION OF ASSIGNED ALTITUDE DEVIATION (DENVER 1984 and 1985 SAMPLES)

APPENDIX A

DISTRIBUTION FAMILY

APPENDIX A

A. Normal Probability Density Function

$$f_A(x,\sigma) = \frac{1}{2\pi\ \sigma} \exp\left[-\frac{1}{2}\frac{x^2}{\sigma^2}\right]$$

$$-\infty < x < \infty$$
$$0 < \sigma < \infty$$

Source: Abramowitz, M., and Stegun, I.A., Handbook of Mathematical Functions, Dover Publications Inc., New York, 1972, p. 930

B. Double-Exponential Probability Density Function

$$f_B(x,\beta) = \frac{1}{2\beta} \exp\left[-\frac{|x|}{\beta}\right]$$

$$-\infty < x < \infty$$
$$0 < \beta < \infty$$

Source: Same as for distributional form A

C. Mixture of Two Double-Exponential Probability Density Functions

$$f_C(x,\beta_1,\beta_2,\alpha) = (1-\alpha)\ f_B(x,\beta_1) + \alpha f_B(x,\beta_2)$$

$$-\infty < x < \infty$$
$$0 < \beta_1 < \beta_2 < \infty$$
$$0 \leq \alpha \leq 1$$

Source: Busch, A., Colamosca, B., "Navigational Systems Requirements Via Collision Risk Model", p. 4

D. Mixture of Two Normal Probability Density Functions

$$f_D(x,\sigma_1,\sigma_2,\alpha) = (1-\alpha)f_A(x,\sigma_1) + \alpha f_A(x,\sigma_2)$$

$$-\infty < x < \infty$$
$$0 < \sigma_1 < \sigma_2 < \infty$$
$$0 \leq \alpha \leq 1$$

E. Power-Exponential Probability Density Function

$$f_E(x,\phi,\beta) = k\phi^{-1} \exp\left[-\tfrac{1}{2}\left|\frac{x}{\phi}\right|^{\left(\frac{2}{1+\beta}\right)}\right]$$

$$\text{where } k^{-1} = \Gamma\left[1+\left(\frac{1+\beta}{2}\right)\right]\cdot 2^{\left[1+\left(\frac{1+\beta}{2}\right)\right]}$$

$$-\infty < x < \infty$$
$$-1 < \beta \leq 1$$
$$0 < \phi < \infty$$

Source: Box, G., and Tiao, G., Bayesian Inference in Statistical Analysis, Addison-Wesley Publishing Company, 1973, pp. 156-158

F. Mixture of Power-Exponential and Double-Exponential Probability Density Functions

$$f_F(x,\phi,\beta_1,\beta_2,\alpha) = (1-\alpha)f_E(x,\phi,\beta_1) + \alpha f_B(x,\beta_2)$$

$$-\infty < x < \infty$$
$$-1 < \beta_1 \leq 1$$
$$0 < \phi < \infty$$
$$0 < \beta_2 < \infty$$
$$0 \leq \alpha \leq 1$$

Source: Rigolizzo, R., "Reduction and Analysis of Mode C Altitude Data Collected at High Altitude Over the Continental United States", U.S. Department of Transportation, DOT/FAA/EM-82/9A, March 1982

G. Mixture of Power-Exponential and Normal Probability Density Functions

$$f_G(x,\sigma,\phi,\beta,\alpha) = (1-\alpha)\, f_A(x,\sigma) + \alpha\, f_E(x,\phi,\beta)$$

$$-\infty < x < \infty$$
$$0 < \sigma < \infty$$
$$0 < \phi < \infty$$
$$-1 < \beta \leq 1$$
$$0 \leq \alpha \leq 1$$

H. Mixture of Two Power-Exponential Probability Density Functions with the same value of β

$$f_H(x,\phi_1,\phi_2,\beta,\alpha) = (1 - \alpha)\ f_E(x,\phi_1,\beta) + \alpha\ f_E(x,\phi_2,\beta)$$

$$\begin{aligned} -\infty &< x < \infty \\ 0 &< \phi_1 < \phi_2 < \infty \\ -1 &< \beta \leq 1 \\ 0 &\leq \alpha \leq 1 \end{aligned}$$

Source: Hsu, D.A., "Further Analysis of Position Errors in Navigation", Journal of Navigation, Volume 3, Number 3, September 1980, pp. 450-474

I. Mixture of Two Power-Exponential Probability Density Functions

$$f_I(x,\phi_1,\beta_1,\phi_2,\beta_2,\alpha) = (1 - \alpha)\ f_E(x,\phi_1,\beta_1) + \alpha\ f_E(x,\phi_2,\beta_2)$$

$$\begin{aligned} -\infty &< x < \infty \\ 0 &< \phi_1 < \phi_2 < \infty \\ -1 &< \beta_1 \leq 1 \\ -1 &< \beta_2 \leq 1 \\ 0 &\leq \alpha \leq 1 \end{aligned}$$