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# Optimization of the Dilation Characteristic on the Control Jurisdiction of an Area Control Facility

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16. Abstract

In an air traffic control environment such as the National Airspace System the control function is based on stereographic representations of aircraft positions in a plane tangent to a sphere with a center collocated with the center of an ellipsoidal model of the geoid. The variation of the dilation (i.e., the discrepancy between the length of an infinitesimal arc on the model surface and the image of the arc in the plane) over the control jurisdiction of an air traffic control facility is one of many factors that adversely affect the ability of the facility to maintain separation standards. Techniques are disclosed for selecting a tangency point and a radius for the spherical support of the stereographic plane that minimize the variation of the dilation over the control jurisdiction about a predetermined constant. The constant can be viewed as a specification of the scale of the map that is the stereographic image in the plane of the portion of the surface of the earth model underlying controlled airspace.

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## EXECUTIVE SUMMARY

In an air traffic control system where aircraft separation is maintained by controlling the trajectories of stereographic representations of aircraft in a plane tangent to a sphere with a center collocated with the center of the earth, dilation is one of many factors that adversely affect system performance. The length of the stereographic image of an arc on the mean sea level surface of the earth (i.e., the surface of an ellipsoid that approximates the geoid) differs from the length of the arc, and this dilation phenomenon is nonuniform in the sense that the lengths of the images of distinct equilength arcs need not be the same (i.e., the length of the image of an arc is a function of both the arc length and the position of the arc on the surface of the earth). As a result, the stereographic representation of the relative separation associated with a pair of aircraft at the same altitude varies with the absolute position of either member of the pair as well as with the relative separation itself. Also, the stereographic image of an aircraft exhibits an acceleration that reflects the change in the dilation with position rather than the physical characteristics of the actual flight trajectory. Indeed, if steps are not taken to minimize the variation of dilation over the portion of the surface of the earth underlying controlled airspace, dilation effects may seriously impact the design of automatic tracking and separation assurance features that are supposed to provide accurate forecasts of future positions of aircraft images in the stereographic plane.

The dilation at any point on the mean sea level surface of the earth is customarily expressed in terms of the ratio of the length of an infinitesimal arc through the point to the length of the stereographic image of the arc. The functional behavior of this measure of dilation over the control jurisdiction of a control facility is governed by the location of the tangency point and the radius of the sphere that supports the stereographic plane. The maximum value of the magnitude of the rate of change of the measure with respect to distance in the floor of the control jurisdiction (i.e., the portion of the surface of the earth underlying the control jurisdiction) is strongly dependent on the latitude and longitude of the tangency point. Since the magnitude of the acceleration of an aircraft image due to dilation tends to increase with the magnitude of the dilation rate, it makes sense to select a latitude-longitude pair for which the maximum value of the magnitude of the rate is small. Depending upon the shape of the floor, this criterion for tangency point selection may result in a latitude-longitude pair corresponding to a point in the floor or outside the floor. In any event, determination of a latitude-longitude pair for the tangency point automatically fixes the ratio of the maximum and minimum values of the dilation on the floor. Subsequent to the selection of the latitude and longitude of the point of tangency the absolute values of these extremums are completely controlled by the radius of the sphere that supports the stereographic

## 1. INTRODUCTION.

In an air traffic control (ATC) environment like the National Airspace System (NAS) aircraft are separated in at least one of the three dimensions of altitude, latitude, and longitude by means of commands transmitted from a ground facility. Aircraft altitude is conveyed to the ground facility by voice communication or transponder replies to radar interrogation messages. The remaining dimensions are obtained indirectly through a projection algorithm that manipulates raw surveillance information (i.e., reported altitude and radar measurements of slant range and azimuth). The objective of the projection algorithm is to provide a point on a plane surface (i.e., the system plane of the control facility) that is a stereographic representation of aircraft latitude and longitude with respect to some ellipsoidal earth model. The control facility attempts to separate aircraft in one or both of the dimensions of latitude and longitude by keeping whatever points are provided by the projection scheme a prescribed distance (e.g., 5 nautical miles (nmi)) apart.

There are two factors, among many others, that adversely affect the ability of the control facility to maintain separation standards. These are projection error (i.e., the difference between the actual stereographic representation of an aircraft and the point representation provided by the projection algorithm) and dilation (i.e., the discrepancy between the distance separating points on the surface of the earth model and the distance between the stereographic representations of these points in the system plane). If the projection error is large then there is some question as to what is being controlled (i.e., latitude and longitude or something else). If it is small then there is no such question. Dilation adversely affects the ability of the control facility to maintain separation standards in the sense that it gives the facility a false impression of the actual distance measured over the surface of the earth model between one location and another that cannot be corrected by a constant scale factor.

Projection errors are pretty much a function of the projection algorithm. On the other hand, dilation is determined by the geometric relationship between the system plane and the earth model. In particular, the system plane is tangent to a conformal sphere (i.e., a sphere with a center collocated with that of the earth model) and dilation is dependent upon the location of the point of tangency and the radius of the sphere. In other words, there are two basic jobs involved in the design of a stereographic projection system in ATC applications. First, the designer must select a conformal sphere and a point of tangency that provide a reasonable dilation characteristic over the control jurisdiction. Second, the designer must provide an algorithm that is capable of converting raw surveillance information into a point on the system plane with little or no projection error. In the case of the Advanced Automation System

$$f(g_c) := 2/[1 + \cos(g_c)] \qquad (1)$$

where $g_c$ is the angle subtended at the center of the earth model by the point of tangency of the system plane and a conformal representation of the point p (i.e., the point on the surface of a conformal sphere at the conformal latitude and longitude of p). Another is the ratio of the radius E of the sphere that supports the system plane to the equatorial radius a of the earth model. The remaining factor is a function h(J) of the geodetic latitude J of p that increases monotonically from 1 to a number just under 1.0034 as the magnitude of the latitude increases from 0° to 90°. Unfortunately, this function is complicated, and as a result, the formula

$$m_p = h(J)(E/a)f(g_c) \qquad (2)$$

is not too useful in the design of ATC projection systems.

Although the function h(J) is somewhat complicated, it is clear that dilation is nearly proportional to the simple function $f(g_c)$, and hence, the shape of the dilation characteristic over the floor of a control jurisdiction is strongly dependent on the parameter $g_c$. As will be seen, there are two aspects of this parameter that are significant from the standpoint of projection system design. One of these is based on the fact that there are infinitely many conformal representations of the same point on the surface of the earth model. The other stems from the fact that the angle subtended at the center of the earth model by two points on the model surface is essentially the same as the angle subtended at the same location by conformal representations of the points.

The angle $g_c$ and the specific conformal sphere of radius E that supports the system plane are independent entities. There are as many conformal representations of a point on the surface of the earth model as there are conformal spheres, and all of these representations lie on the same radial line segment emanating from the center of the earth. Consequently, there are infinitely many conformal representations of the floor of a control jurisdiction, and regardless of whether the point p is a member of the floor, $g_c$ is the unique angle subtended at the center of the earth model by the point of tangency and any conformal representation of p. Also, if q is the earth model representation of the point of tangency (i.e., the location on the model surface specified by the geodetic latitude and longitude of the tangency point) then the same angle is subtended at the center of the earth by any conformal representation of the point p and any conformal representation of q, including the tangency point itself. For example, the angle $g_c$ is the same as the angle subtended at the center of the earth by the conformal representations of p and q on the unit sphere (i.e., the conformal sphere characterized by a radius of 1 in whatever units the dimension of length is expressed). This concept can be exploited in the selection of the tangency point.

table 1. Approximations of angles like $g_c$ by parameters of the type d/60 and g are useful in the selection of a near optimal radius for the conformal sphere that supports the system plane.

TABLE 1. ESTIMATION ACCURACIES OF d/60 AND g.

| d (nmi) | g (deg) | $B_1(d)$ ($10^{-7}$) | $B_2(g)$ ($10^{-7}$) |
|---|---|---|---|
| 60 | 1 | 3.2 | 0.4 |
| 120 | 2 | 11.9 | 0.9 |
| 180 | 3 | 26.0 | 1.3 |
| 240 | 4 | 45.6 | 1.7 |
| 300 | 5 | 70.8 | 2.2 |
| 600 | 10 | 279.4 | 4.3 |
| 900 | 15 | 627.5 | 6.5 |
| 1200 | 20 | 1117.7 | 8.7 |
| 1500 | 25 | 1753.9 | 10.9 |
| 1800 | 30 | 2541.3 | 13.2 |
| 2100 | 35 | 3486.4 | 15.5 |
| 2400 | 40 | 4597.3 | 17.9 |

## 3. ESTIMATION OF DILATION.

The dilation at any point in the floor of the control jurisdiction is closely approximated by the function

$$m(g_c) = k(E/a)f(g_c) \tag{5}$$

where k is the arithmetic mean of upper and lower bounds in the interval from 1 to 1.0034 on the set of values of the function h(J) associated with the geodetic latitudes of the points in the floor. Indeed, if $h_1$ is a lower bound, $h_2$ is an upper bound, and

$$k = (h_1 + h_2)/2 \tag{6}$$

then, as shown in appendix D, the dilation at any point in the floor must satisfy the inequality

$$\mathrm{mag}[m_p - m(g_c)] \leq [(h_2 - h_1)/(h_1 + h_2)]m(g_c). \tag{7}$$

Thus, if the bounds are 1 and 1.0034 (i.e., k is 1.0017) then the use of $m(g_c)$ as an esimate of the dilation at any point on the surface of the earth model results in an estimation error that is at most 0.17 percent of the estimate. Table 2 implies that tighter bounds than these apply to smaller regions of the surface of the earth. Consequently, as demonstrated in examples 1 and 2, greater estimation accuracies can be realized over the floor of an ACF control jurisdiction.

$$\mathrm{mag}[u_p(n) - v(n,g_c)] \leq [(h_m - h_1)/(h_1 + h_m)][v(n,g_c) + n] \quad (10)$$

for every point p in the floor of the control jurisdiction. As will be seen, it is possible to keep $v(n,g_c)$ below 0.0352n on the floor of an ACF control jurisdiction that fulfills the size requirements of the AAS specification. Stated another way, if the system plane point of tangency and the radius of the conformal sphere supporting the system plane are properly selected then $v(n,g_c)$ will differ from the magnitude of the difference between the dilation and n by at most

$$V(n) = 1.0352[(h_m - h_1)/(h_1 + h_m)]n. \quad (11)$$

Since $h_m - h_1$ is at most 0.0034 and $h_1 + h_m$ is never less than 2 it follows that V(n) is at most 0.176 percent of n. In practice, the value of V(n) can be made much smaller than this. For instance, V(n) is 0.000296n in example 1 and it is 0.000115n in example 2.

## 4. SELECTION OF THE SPHERICAL SUPPORT RADIUS.

Assuming that the earth model representation of the point of tangency is known (i.e., the angle $g_c$ is defined) the estimate $m(g_c)$ can be viewed as a function of the parameters E and $g_c$, and it is clear that to each positive number that might be used as the radius of the spherical support for the system plane there corresponds some maximum value, other than 0, of $v(n,g_c)$ on the floor of the control jurisdiction. Fortunately, among all such numbers it is easy to locate one for which the maximum is least. As shown in appendix E, this optimal radius is

$$E_o(n,F_c,G_c) = 2n(a/k)/[f(F_c) + f(G_c)] \quad (12)$$

where $F_c$ and $G_c$ represent the minimum and maximum angles, respectively, subtended at the center of the earth model by the system plane point of tangency and a point within any conformal representation of the floor. Using r to represent the ratio of $f(G_c)$ to $f(F_c)$, the formula

$$v_o(n,r) = n[(r - 1)/(r + 1)] \quad (13)$$

supplies the corresponding maximum value of the magnitude of the difference between n and the estimate of dilation on the floor of the control jurisdiction. If the earth model representation of the point of tangency is located in the floor then $f(F_c)$ is 1 (i.e., $F_c$ is 0) and $v_o(n,r)$ takes on the value

$$v_o(n,f(G_c)) = nw(G_c) \quad (14)$$

where

$$w(G_c) = [f(G_c) - 1]/[f(G_c) + 1]. \quad (15)$$

column upper bounds the magnitude of the difference between $w(G_c)$ and $w(G)$. As demonstrated by the table, the estimate of the maximum deviation of the dilation from n associated with the optimal radius is not significantly different from the deviations associated with the suboptimal radii. In other words, if the great circle distances $D_G$ and $D_F$ are available then there is little reason for one to be reluctant about employing $E_G(n, D_F/60, D_G/60)$ as the radius of the spherical support for the

TABLE 3. BOUNDS ON $mag[w(G_c) - w(G)]$ AND $mag[w(G_c) - w(D_G/60)]$

| $D_G$ (nmi) | G (deg) | $w(D_G/60)$ or $w(G)$ ($10^{-4}$) | $B_G(D_G)$ ($10^{-4}$) | $B_A(G)$ ($10^{-4}$) |
|---|---|---|---|---|
| 300 | 5.00 | 9.5 | 0.0354 | 0.0011 |
| 600 | 10.00 | 38.1 | 0.1397 | 0.0022 |
| 900 | 15.00 | 85.9 | 0.3137 | 0.0032 |
| 1200 | 20.00 | 153.1 | 0.5587 | 0.0043 |
| 1500 | 25.00 | 239.8 | 0.8764 | 0.0055 |
| 1800 | 30.00 | 346.5 | 1.2689 | 0.0066 |
| 1815 | 30.25 | 352.4 | 1.2906 | 0.0066 |
| 2100 | 35.00 | 473.5 | 1.7390 | 0.0077 |
| 2400 | 40.00 | 621.2 | 2.2892 | 0.0089 |

system plane. Likewise, $E_G(n,F,G)$ can be used as the radius of the supporting sphere whenever the angles G and F are available. Appendix G discloses the procedure used to generate the bounds in table 3.

The parameters $D_F$, $D_G$, F, and G may not be readily available. For example, there is always some inherent distortion associated with any planar representation of a portion of the surface of the earth, and so it is not possible to obtain an exact determination of the distances $D_F$ and $D_G$ from a standard map illustrating the boundary of the floor of the control jurisdiction. On the other hand, the latitude and longitude of each radar in the network of radars supporting the surveillance function of an ACF will most certainly be known. This information together with the latitude and longitude of the tangency point can be used to compute the degree measures of the minimum and maximum angles $A_{min}$ and $A_{max}$ subtended at the center of the earth by the earth model representation of the point of tangency and a location on the model surface corresponding to the latitude and longitude of a member of the radar network. Also, it is most likely that some reasonable upper bound S on the ranges of the network radars will be available. For example, the maximum effective range of the search radars employed in NAS is often quoted as being 200 nmi. Consequently, the sum of the degree measure of $A_{max}$ and the ratio of S in nmi to 60 can be viewed as a reasonable estimate of the degree measure of G. In like manner, the angle F can be taken to be the maximum of the numbers 0 and $A_{min} - S/60$. In those cases where the members of the radar network are not endowed with the same effective range, this technique can be extended to provide

of motion of the representation, and the magnitude of the acceleration is in the neighborhood of the product of 1/60, pi/180, s(d/60), and the square of c.

Minimization of the parameter r does not guarantee that the maximum value of the slope of the dilation estimate on the floor of the control jurisdiction will be a minimum. The maximum value of the estimate on the floor associated with the optimal radius is the sum of $v_\omega(n,r)$ and n. Thus,

$$s_\omega(n,r,G_\omega) = [v_\omega(n,r) + n]\tan(G_\omega/2) \tag{18}$$

is the maximum value of the slope of the estimate on the floor associated with the optimal radius. Since $G_\omega$ is not necessarily a monotone increasing function of r there is little reason to believe that the best position for the point of tangency is a location that renders r a minimum.

A good position for the tangency point is any location that minimizes the maximum angle $G_\omega$ subtended by the point of tangency and a point in a conformal representation of the floor. Regardless of the location of the tangency point, the parameter r is at least 1 and at most $f(G_\omega)$. As a result,

$$s_\omega(n,f(G_\omega),G_\omega) = n[w(G_\omega) + 1]\tan(G_\omega/2) \tag{19}$$

bounds the maximum slope $s_\omega(n,r,G_\omega)$ from above and

$$s_\omega(n,1,G_\omega) = n\tan(G_\omega/2) \tag{20}$$

bounds it from below. Since $w(G_\omega)$ is an increasing function of $G_\omega$ the same is also true for the magnitude of the difference between these bounds. Also, (19) implies that the maximum slope $s_\omega(n,r,G_\omega)$ differs from the lower bound by at most $100w(G_\omega)$ percent of that bound. Finally, as shown in the previous section, $nw(G_\omega)$ upper bounds $v_\omega(n,r)$. Thus, there is ample reason to view the function $w(G_\omega)$ as a measure of goodness of the location of the point of tangency, and much can be gained by selecting a location for the tangency point for which the angle $G_\omega$ is least.

Table 3 illustrates that the sensitivity of the measure $w(G_\omega)$ to the position of the earth model representation of the point of tangency is not so critical that one needs to resort to powerful optimization techniques in order to find an appropriate location for the tangency point. For example, $w(D_\omega/60)$ increases from 0.02398 to 0.04735 as $D_\omega$ increases from 1500 nmi to 2100 nmi. Thus, the rate of change of the measure per unit change in distance is roughly 0.00004 $nmi^{-1}$ when the largest great circle distance separating a point in the floor of the control jurisdiction from the earth model representation of the point of tangency is in the neighborhood of 1800 nmi. Consequently, an 87 nmi movement of q will result in a change in the measure $w(D_\omega/60)$ of about 10 percent. In fact, the value of $w(D_\omega/60)$ increases

longitude of $q_c$ when the number of points in the set R is infinite. On the other hand, if R is finite then the determination of the latitude and longitude of $q_c$ reduces to a linear programming problem. The following algorithm is a prescription for setting up the linear programming problem and acquiring the conformal latitude and longitude of $q_c$ from the problem solution. A detailed development of the algorithm is provided in appendix I.

Algorithm. Tangency point selection via linear programming.

Input data for the algorithm consists of n pairs of numbers $(L_1, M_1),\ldots,(L_n, M_n)$ where $L_k$ and $M_k$ are the longitude and conformal latitude, respectively, of the kth ($k = 1,\ldots,n$) member $p_k$ of a finite set R of n points on the surface of the earth model that represents the extent of the floor of the control jurisdiction. In effect, the pair $(L_k, M_k)$ represents the spherical coordinates of the conformal representation of $p_k$ on the unit sphere. It is assumed that the conformal representation of the entire set R on the surface of the unit sphere is a subset of a hemisphere. Since the floor of a control jurisdiction meeting the AAS design limit on the size of the coverage region of an ACF is much less than half the surface of the earth model, this assumption is satisfied in any case where R is a subset of any portion of the model surface that qualifies as the floor of a control jurisdiction in the context of the AAS specification.

a. Construct a 3xn matrix B such that the elements of the kth column of the matrix are the Cartesian coordinates of the kth element of the set R on the unit sphere. Thus, the 3x1 matrix

$$\underline{b}_k = \begin{vmatrix} \cos(M_k)\cos(L_k) \\ \cos(M_k)\sin(L_k) \\ \sin(M_k) \end{vmatrix} \tag{22}$$

is the kth column of B.

b. Construct the nxn symmetric matrix A that is the result of premultiplying the matrix B by its transpose.

c. Letting $\underline{1}$ represent the nx1 matrix with identical elements equal to 1, $\underline{1}^T$ the transpose of $\underline{1}$, and $\underline{0}$ the nx1 matrix with identical elements equal to 0, find an nx1 matrix $\underline{x}$ and a scalar r such that the scalar is a minimum subject to the constraints that r and all the elements of $\underline{x}$ are non-negative,

$$A\underline{x} - r\underline{1} = \underline{0}, \tag{23}$$

and

$$\underline{1}^T\underline{x} = 1. \tag{24}$$

This is a linear programming problem in n + 1 variables (i.e., r

closed curve constructed on the surface of a specific conformal sphere. The radius of the sphere is the arithmetic mean of the equatorial and polar radii of the earth model (i.e., 3,438.146 nmi) and the curve consists of four great circle arcs each of which is exactly 2500 nmi in length. As shown in appendix J, the center of the smaller of the two regions bounded by the four arcs is separated from each vertex of the boundary by a great circle distance of 1,811.774 nmi. If the AAS design limit is interpreted to be the radial projection of the smaller region onto the surface of the earth model then the limit corresponds to the somewhat fuzzy, but nevertheless realistic, notion of a region of the model surface having a square-like perimeter that is approximately 10,000 nmi in length and a centrally located point that is roughly 1,812 nmi from each vertex of the perimeter. In fact, as shown in appendix J, adjacent vertices of the perimeter are separated by a great circle distance that is between 2,495.802 nmi and 2,504.198 nmi, and the great circle distance between the centrally located point and any vertex of the perimeter is not less than 1,808.732 nmi nor more than 1,814.817 nmi. Needless to say, the maximum great circle distance separating the centrally located point from any other point in the square-like region cannot exceed the upper bound of 1,814.817 nmi.

Some very definite implications with respect to dilation are associated with a design limit that is a region of the surface of the earth model with a centrally located point that is separated from any other point in the region by a great circle distance less than or equal to a prescribed maximum $D_L$ that is itself known to be less than 1,815 nmi. Specifically, the floor of any ACF control jurisdiction commissioned in the AAS can be envisioned as being embedded in the design limit. Thus, regardless of the location of the point of tangency and the size of the radius of the sphere that supports the system plane, the euclidean norm of the difference between the dilation and the design constant n on the floor of the control jurisdiction will not exceed the maximum value of the norm of the difference on the design limit. Consequently, there is no reason why the norm of the difference on the floor should ever exceed the maximum value of the norm on the limit in the case where the point of tangency is centrally located with respect to the limiting region and the size of the conformal radius is the subobtimal value $E_\omega(n,0,D_L/60)$. Since $D_L$ is less than 1815 nmi, it follows from table 3 that this maximum norm is less than 0.0352n. As a result, it can be concluded that the magnitude of the difference between the dilation and n on the floor of a control jurisdiction falling within the AAS design limit can be kept below 0.0352n by (1) assigning a location to the point of tangency such that among all possible positions for the tangency point the maximum angle subtended at the center of the earth by the assigned location and a point in a conformal representation of the floor is essentially minimal and (2) using either one of the numbers $E_\omega(n,D_F/60,D_G/60)$ and $E_\omega(n,F,G)$ as the radius of the conformal support for the system plane.

most 1.0352n and the minimum dilation is at least 0.9648n. In other words, if the tangency point and the spherical support radius are selected in accord with the techniques outlined in this report then the dilation will deviate from the dilation design constant by at most 3.52 percent of the constant over the control jurisdiction of any ACF commissioned in the AAS.

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## APPENDIX B. ANGLE ESTIMATION ERROR

The angle g subtended at the center of the earth model by points p and q on the model surface is a fairly good estimate of the angle $g_c$ subtended at the same location by conformal representations of the points. Since the longitude of a point on the earth model is the same as the longitude of the conformal representation of the point, the difference between the two angles is due solely to the fact that the geocentric and conformal latitudes of the same point on the model surface are not necessarily the same. In fact,

$$\cos(g) = \sin(x)\sin(y) + A[\cos(x)\cos(y)] \qquad \text{(B-1)}$$

where x and y are the geocentric latitudes of points p and q, respectively, and A is the cosine of the difference between the longitudes of the two points. The corresponding formula for $\cos(g_c)$ can be obtained from the expression for cos(g) by replacing x with the conformal latitude $x_c$ of the point p and y with the conformal latitude $y_c$ of the point q. Since the geocentric and conformal latitudes of a point on the surface of the earth differ by at most $0.000141^\circ$ it is apparent that the difference between the angles g and $g_c$ is small.

As already shown in appendix A, the magnitude of the difference between the degree measure of g and the product of 1/60 and the great circle distance d in nmi separating the points p and q is at most 0.18 percent of d/60, and so the ratio d/60 should provide a good estimate of $g_c$. In fact, since $g_c - d/60$ is the same as the sum of $g_c - g$ and $g - d/60$ it follows that

$$\text{mag}(g_c - d/60) \leq \text{mag}(g_c - g) + \text{mag}(g - d/60). \qquad \text{(B-2)}$$

As a result, $\text{mag}(g_c - d/60)$ is bounded above by the sum of the bound $0.0018(d/60)$ on $\text{mag}(g - d/60)$ and any upper bound on the degree measure of $\text{mag}(g_c - g)$.

A numerical bound on the difference between the angles g and $g_c$ can be derived in terms of the gradient vector of the function arc cos(g) with respect to the arguments x and y. Specifically, if C is an upper bound on the magnitude of the gradient vector for all possible values of x, y, and A then

$$\text{mag}(g - g_c) \leq C[(x - x_c)^2 + (y - y_c)^2]^{1/2}. \qquad \text{(B-3)}$$

Since $0.000141^\circ$ is an upper bound on the degree measures of $\text{mag}(x - x_c)$ and $\text{mag}(y - y_c)$ it follows that the degree measure of the magnitude of the difference between g and $g_c$ cannot exceed the bound

$$B = C[2(0.000141)^2]^{1/2}. \qquad \text{(B-4)}$$

It only remains to provide a numerical value for C.

## APPENDIX C. EFFECT OF ANGLE ESTIMATION ERROR ON THE COMPUTATION OF DILATION

If x is the degree measure of an angle that is less than 180°, y is the degree measure of a like angle, and x and y differ at most by a positive number p(x) such that the sum of x and p(x) is less than 180° then the ratio f(x) of 2 to 1 + cos(x) must satisfy the inequality

$$\mathrm{mag}[f(y)/f(x) - 1] \leq B(x,p(x)) \qquad \text{(C-1)}$$

where

$$B(x,p(x)) = f(x + p(x))/f(x) - 1. \qquad \text{(C-2)}$$

Indeed, f(x) is a positive function of the degree measure x on the interval I of nonnegative numbers less than 180, and so

$$\mathrm{mag}[f(y)/f(x) - 1] = [1/f(x)]\mathrm{mag}[f(y) - f(x)]. \qquad \text{(C-3)}$$

Also, the first and second derivatives of f(x) with respect to x are nonnegative functions on I. Thus,

$$\mathrm{mag}[f(y) - f(x)] \leq f(x + p(x)) - f(x) \qquad \text{(C-4)}$$

and the bound B(x,p(x)) follows directly upon dividing both sides of the inequality by f(x).

As shown in appendix B, the great circle distance d in nmi separating points p and q on the surface of the earth model and the degree measure of the angle $g_c$ subtended at the center of the earth by conformal representations of p and q satisfy the inequality

$$\mathrm{mag}[d/60 - g_c] \leq p_1(d/60) \qquad \text{(C-5)}$$

where

$$p_1(d/60) = 0.0018(d/60) + 0.000282. \qquad \text{(C-6)}$$

If the sum of d/60 and $p_1(d/60)$ is less than 180° (i.e., d is less than 10780.58 nmi) then

$$B_1(d) = B(d/60,p_1(d/60)) \qquad \text{(C-7)}$$

is an upper bound of the magnitude of the difference between 1 and the ratio of $f(g_c)$ to f(d/60).

In appendix B it is shown that the degree measures of $g_c$ and the angle g subtended at the center of the earth by the points p and q differ at most by

$$p_2(g) = 0.000282. \qquad \text{(C-8)}$$

## APPENDIX D. ACCURACY OF THE DILATION ESTIMATE

Suppose that $x_1$ and $x_2$ are numbers such that the former is less than the latter, $x_0$ is halfway between the limits $x_1$ and $x_2$, t is somewhere between the limits, and n is any number inside or outside of the interval extending from $x_1$ to $x_2$. If both $x_0$ and t lie on the same side of n then the distance separating mag($x_0$ − n) from mag(t − n) is the same as the distance separating $x_0$ from t. Otherwise, the distance separating mag($x_0$ − n) from mag(t − n) cannot exceed half the distance separating $x_1$ from $x_2$. In any case, the distance separating $x_0$ and t can never be greater than half the distance between $x_1$ and $x_2$. In other words, the inequality

$$\mathrm{mag}[\mathrm{mag}(x_0 - n) - \mathrm{mag}(t - n)] \leq (x_2 - x_1)/2 \qquad \text{(D-1)}$$

is valid for any number n.

If factor h(J) in (2) is bounded below by $h_1$ and above by $h_2$ then the dilation $m_p$ lies between the limits

$$x_1 = [h_1 m(g_c)]/k \qquad \text{(D-2)}$$

and

$$x_2 = [h_2 m(g_c)]/k. \qquad \text{(D-3)}$$

If, in addition, k is the arithmetic mean of $h_1$ and $h_2$ then $m(g_c)$ is halfway between the limits, and the distance separating mag($m_p$ − n) and mag[$m(g_c)$ − n] cannot exceed

$$(x_2 - x_1)/2 = (1/2)[(h_2 - h_1)/k]m(g_c). \qquad \text{(D-4)}$$

Moreover, if $m(g_c)$ is not less than n then it is the same as the sum of mag[$m(g_c)$ − n] and n. Otherwise, the sum exceeds $m(g_c)$. In any event,

$$(x_2 - x_1)/2 \leq (1/2)[(h_2 - h_1)/k](\mathrm{mag}[m(g_c) - n] + n) \qquad \text{(D-5)}$$

for any number n, including Ø.

## APPENDIX F. ESTIMATION OF ANGULAR EXTREMUMS

Let ang(x,y) represent the degree measure of the angle subtended at the center of the earth by any two points x and y. Also, let G represent the least upper bound of the set A of all numbers of the form ang(q,p) where q is the earth model representation of the point of tangency and p is a point in the floor K of the control jurisdiction. If there is a point s in the floor such that ang(q,s) is the same as G then G is the maximum angle subtended at the center of the earth by q and a point in the floor. Otherwise, the notion of a maximum angle is meaningless and it is necessary to resort to the concept of a least upper bound. In any event, an inequality identical to (3) can be established for G and the least upper bound $G_c$ of the set $A_c$ of all numbers of the form ang($q_c$,$p_c$) where $q_c$ is the tangency point and $p_c$ is a member of the conformal representation $K_c$ on the unit sphere of the floor K. Likewise, it can be shown that the same inequality applies to the greatest lower bounds F and $F_c$ of the sets A and $A_c$, respectively.

The mapping ang(q,p) is a continuous function of p on the surface of the earth model, and the closure of K (i.e., the set of all points that are in the floor and the boundary of the floor) is a bounded closed set in euclidean 3-space. Thus, ang(q,p) attains absolute minimum and maximum values on the closure of the floor, and so there exists a point t in the closure such that ang(q,t) is G. The inequality (3) implies that

$$\text{ang}(q_c, t_c) \geq G - z \qquad \text{(F-1)}$$

where $t_c$ is the conformal representation on the unit sphere of t and the symbol z represents 0.000282°. Since the relationship between geodetic and conformal latitudes is a homeomorphism (i.e., a bijective function for which both the function and its inverse are continuous) the same is true of the transformation T that maps the surface of the earth model into the conformal representation of the model surface on the unit sphere. Thus, the closure of the conformal representation $K_c$ of the floor is the image of the closure of the floor under the mapping T. Consequently, $t_c$ is a member of the closure of $K_c$, and it follows that

$$\text{ang}(q_c, t_c) \leq G_c. \qquad \text{(F-2)}$$

Clearly, (F-1) contradicts (F-2) under the assumption that G exceeds the sum of $G_c$ and z. As a result, it must be concluded that

$$G \leq G_c + z. \qquad \text{(F-3)}$$

The inequality

$$F \geq F_c - z \qquad \text{(F-4)}$$

$$b(pi/180)G \leq dist(q,t) \leq a(pi/180)G \qquad (F-11)$$

where a is the equatorial radius of the earth model in nmi and b is the polar radius in nmi. Since $D_G$ is at least dist(q,t) and ang(q,s) cannot exceed G, it follows that

$$b(pi/180)G \leq D_G \leq a(pi/180)G. \qquad (F-12)$$

Following the argument in appendix A, it is now a simple matter to show that

$$mag(D_G/60 - G) \leq 0.0018(D_G/60) \qquad (F-13)$$

and

$$mag(60G - D_G) \leq 0.0018(60G). \qquad (F-14)$$

In like manner, it can be shown that $mag(D_F/60 - F)$ cannot exceed 0.18 percent of $D_F/60$ and $mag(60F - D_F)$ is never greater than 0.18 percent of 60F.

$$B_4(G) = C(G, p_{\equiv}(G)) \qquad (G\text{-}8)$$

whenever the sum of G and $p_{\equiv}(G)$ does not exceed 124.16°.

to the norm of $\underline{w}$.

The conformal representation $R_c$ of the set R is closed and bounded. Specifically, the norm of each element of $R_c$ is bounded above by 1, and so $R_c$ is certainly bounded. Also, the bijection T and its inverse $T^{-1}$ are continuous functions. Consequently, since R is closed, the image of R under the transformation T (i.e., $R_c$) is itself closed.

Among all the vectors in the convex hull $H(R_c)$ of the set $R_c$ there exists one, and only one, vector $\underline{w}$ with a norm that is less than the norm of any other vector in the hull. The assumption that there is a unit vector $\underline{z}$ and a positive number t such that

$$\underline{y} \text{ in } R_c \text{ implies } (\underline{z},\underline{y}) \geq t \qquad \text{(H-1)}$$

guarantees that $R_c$, and hence, the convex hull of $R_c$, is a subset of the convex set

$$J = \{\underline{y}\text{: norm}(\underline{y}) \leq 1 \text{ and } (\underline{z},\underline{y}) \geq t\}. \qquad \text{(H-2)}$$

Since t exceeds Ø, the origin of $E^3$ is not a member of J. Also, since $R_c$ is closed and bounded, the set $H(R_c)$ is closed and bounded. Thus, by virtue of a well known theorm on the minimum distance to a convex set (Luenberger, D., Optimization by Vector Space Methods, John Wiley, 1969, page 69) there exists one, and only one, vector $\underline{w}$ in the convex hull of $R_c$ such that

$$\underline{y} \text{ in } R_c \text{ implies } (\underline{w},\underline{y}) \geq [\text{norm}(\underline{w})]^2, \qquad \text{(H-3)}$$

and the norm of $\underline{w}$ is strictly less than the norm of any other member of $H(R_c)$.

If $\underline{x}$ is a unit vector then there exists a vector $\underline{s}(\underline{x})$ in $R_c$ such that

$$\underline{y} \text{ in } R_c \text{ implies } (\underline{x},\underline{s}(\underline{x})) \leq (\underline{x},\underline{y}), \qquad \text{(H-4)}$$

and $(\underline{x},\underline{s}(\underline{x}))$ is a continuous function of $\underline{x}$ on the set U of all unit vectors. As a result, there exists a unit vector $\underline{x}_o$ and a member $\underline{y}_o$ of $R_c$ such that

$$(\underline{x}_o,\underline{y}) \geq (\underline{x}_o,\underline{y}_o) \geq \min\{(\underline{x},\underline{y})\text{: } \underline{y} \text{ in } R_c\} \qquad \text{(H-5)}$$

for every unit vector $\underline{x}$ and every member $\underline{y}$ of $R_c$. The existance of $\underline{s}(\underline{x})$ is guaranteed by the fact the inner product $(\underline{x},\underline{y})$ is a continuous function of $\underline{y}$ and the set $R_c$ is both closed and bounded (i.e., a continuous function attains an absolute minimum and an absolute maximum on a closed and bounded set in $E^3$). Also, like $R_c$, the set U is closed and bounded. Thus, the product set $U \times R_c$ is a closed and bounded subset of the product space $E^3 \times E^3$. Since the inner product of two vectors is a continuous function on this space, the inner product is uniformly continuous on $U \times R_c$. As a result, the function $(\underline{x},\underline{s}(\underline{x}))$ is

## APPENDIX I. A LINEAR PROGRAMMING APPROACH TO TANGENCY POINT SELECTION

In this appendix it is assumed that the extent of the floor of the control jurisdiction is fairly represented by a finite set R of n points on the surface of the earth model. For example, each member of R might be a location on the model surface defined by the geodetic latitude and longitude of a radar or it might consist of a group of points more or less uniformly scattered over the floor. In addition, it is assumed that there exists at least one unit vector $\underline{z}$ such that the angle subtended at the center of the earth by $\underline{z}$ and any one of the n members of the conformal representation $R_c$ of R on the unit sphere is less than 90°. As pointed out in appendix H, this assumption is satisfied in any case where R fairly represents the extent of the floor of a control jurisdiction meeting the AAS design limit on the size of the coverage region of an ACF. Also, as established in appendix H, the closed nature of R and the existance of the vector $\underline{z}$ imply that the optimal tangency point is merely a scalar multiple of the minimum norm vector $\underline{w}$ in the convex hull $H(R_c)$ of $R_c$. Consequently, the longitude of $\underline{w}$ is the longitude of the earth model representation $q_o$ of the optimal tangency point, and the geocentric latitude of $\underline{w}$ is the conformal latitude of $q_o$. As will be seen, this appendix discloses a practical technique for finding $\underline{w}$, and hence, $q_o$, in terms of the solution of a linear programming problem.

The set $R_c$ is represented by a 3xn matrix B. The kth column of B is a function of the longitude $L_k$ and the conformal latitude $M_k$ of the kth member of the set R that describes the extent of the floor of the control jurisdiction. As indicated in figure I-1, the center of the earth is viewed as the origin of a Cartesian coordinate system with axes 1,2, and 3 where axis 3 coincides with the polar axis of the earth and the positive direction along axis 3 is from the South Pole to the North Pole. The element $b_{jk}$ in the jth row and kth column of the matrix B is defined in terms of the formulas

$$b_{1k} = \cos(M_k)\cos(L_k), \qquad \text{(I-1)}$$

$$b_{2k} = \cos(M_k)\sin(L_k), \qquad \text{(I-2)}$$

and

$$b_{3k} = \sin(M_k). \qquad \text{(I-3)}$$

Thus, the kth column of the matrix B is a 3x1 matrix $\underline{b}_k$ with elements $b_{1k}$, $b_{2k}$, and $b_{3k}$.

The mimimum norm member of the convex hull $H(R_c)$ of $R_c$ is a 3x1 matrix obtained by postmultiplying B by an nx1 probability matrix. An nx1 matrix $\underline{x}$ is a probability matrix if all components of the matrix are non-negative (denoted by $\underline{x} \geq 0$) and a sum of 1 is obtained when the transpose $\underline{x}^T$ of $\underline{x}$ is

postmultiplied by the nx1 vector $\underline{1}$ with identical components equal to 1. Since the convex hull of a finite set is just the collection of all convex linear combinations of the elements of the set, $H(R_e)$ is equivalent to the set of all 3x1 matrices of the form $B\underline{x}$ such that $\underline{x} \geq \emptyset$ and $\underline{x}^T\underline{1} = 1$.

Letting A represent the nxn symmetric semipositive definite matrix formed by premultiplying B by its transpose $B^T$, the minimum norm element of the convex hull of $R_e$ is a 3x1 matrix of the form $B\underline{x}$ where $\underline{x}$ is any nx1 matrix that minimizes the function

$$G(\underline{x}) = \underline{x}^T A \underline{x} \qquad \text{(I-4)}$$

subject to the constraints

$$\underline{x}^T\underline{1} - 1 = \emptyset \qquad \text{(I-5)}$$

and

$$\underline{x} \geq \emptyset. \qquad \text{(I-6)}$$

The constraints merely restrict $\underline{x}$ to the set of nx1 probablity matrices. The function $G(\underline{x})$ is the result of premultiplying the 3x1 matrix $B\underline{x}$ by its transpose, and so it is just the square of the norm of $B\underline{x}$.

Every solution of the constrained minimization problem is a probability matrix $\underline{x}$ satisfying the equation

$$A\underline{x} - r\underline{1} = \underline{\emptyset} \qquad \text{(I-7)}$$

where r is a scalar and $\underline{\emptyset}$ is the nx1 null matrix (i.e., the elements of $\underline{\emptyset}$ are identical and equal to $\emptyset$). This result follows directly from a straight forward application of the generalized method of Lagrange multipliers to the problem of minimizing $G(\underline{x})$ subject to the constraints (I-5) and (I-6).

Any solution of the linear programing problem in n + 1 variables of minimizing the linear functional r subject to the constraints

$$\left|\begin{array}{c|c} A & \underline{1} \\ \hline \underline{1}^T & \emptyset \end{array}\right| \left|\begin{array}{c} \underline{x} \\ \hline -r \end{array}\right| = \left|\begin{array}{c} \underline{\emptyset} \\ \hline 1 \end{array}\right| \qquad \text{(I-8)}$$

and

$$\left|\begin{array}{c} \underline{x} \\ \hline r \end{array}\right| \geq \emptyset. \qquad \text{(I-9)}$$

is a solution to the problem of minimizing $G(\underline{x})$ subject to the constraints (I-5) and (I-6). The scalar r in equation (I-7) is merely the square of the norm of $B\underline{x}$. This fact can be verified

## APPENDIX J. SPHERICAL SQUARES

If x is a positive number not greater than 90 then there exists a spherical square on the surface of a conformal sphere in the sense that it is possible to select four distinct points $p_0$, $p_1$, $p_2$, and $p_3$ (i.e., the vertices of the square) on the surface of the sphere such that x is the degree measure of the angle subtended at the center of the earth by any pair of adjacent vertices (i.e., vertices $p_{m(i)}$ and $p_{m(i+1)}$ where i is any nonnegative integer and m(i) represents the integer i modulo 4). The spherical coordinates of a point on the sphere consist of the radius of the sphere, the latitude of the point, and the longitude of the point. Let x/2 represent the latitude of $p_1$ and $p_2$, -(x/2) the latitude of $p_0$ and $p_3$, L the longitude of $p_2$ and $p_3$, and -L the longitude of $p_0$ and $p_1$. The degree measure A of the angle subtended at the center of the earth by $p_1$ and $p_2$ is the same as the degree measure of the angle subtended at the same location by $p_0$ and $p_3$. The cosine of this angle is just the inner product of the direction cosine vectors of $p_1$ and $p_2$. Thus,

$$\cos(A) = \cos^2(x/2)[2\cos^2(L) - 1] + \sin^2(x/2), \qquad \text{(J-1)}$$

and so cos(A) decreases from 1 to -cos(x) as L increases from 0° to 90°. Since x is not greater than 90° it follows that A increases from 0° to at least 90° as L moves from 0° to 90°. Consequently, there exists a degree measure $L_0$ not greater than 90° such that A is x when L takes on the value $L_0$, and (J-1) implies that

$$\cos^2(L_0) = \cos(x)/\cos^2(x/2). \qquad \text{(J-2)}$$

In other words, the points $p_0$, $p_1$, $p_2$, and $p_3$ become the vertices of a spherical square when L is $L_0$.

There is a point t on the surface of the sphere associated with the spherical square defined by the vertices $p_0$,$p_1$,$p_2$, and $p_3$ that is centrally located with respect to the square in the sense that the degree measure B of the angle subtended at the center of the earth by t and a vertex of the square is the same for all vertices of the square, the cosine of B is the square root of the cosine of the angle subtended at the center of the earth by adjacent vertices, and the angle subtended at the center of the earth by t and any point p on the shortest great circle arc connecting adjacent vertices is at most B. Let t be the point on the sphere corresponding to 0° latitude and 0° longitude. From considerations of symmetry it is apparent that the degree measure B of the angle subtended at the center of the earth by t and a vertex is the same for all vertices. Also, the cosine of B is just the inner product of the direction cosine vectors of t and any one of the vertices, and so

$$\cos(B) = \cos(x/2)\cos(L_0) = [\cos(x)]^{1/2}. \qquad \text{(J-3)}$$