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Evaluation of a Projection Algorithm for the Stereographic Representation of Aircraft in an Air Traffic Control System

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16. Abstract

In an air traffic control system such as the National Airspace System (NAS) horizontal control is effected from a ground facility through the separation of aircraft in at least one of the dimensions of latitude and longitude. Ideally, these dimensions are obtained indirectly in real time by means of a projection algorithm that maps raw surveillance information into the stereographic representation of aircraft latitude and longitude on a plane surface. In practice, there is always some processing delay and some difference between the algorithm output and the actual stereographic representation of aircraft position. In addition, the accuracy is affected by the orientation of the plane relative to the control jurisdiction of the facility. While the projection algorithm currently employed in NAS is capable of timely delivery of output due to its structural simplicity, it cannot meet the accuracy requirement of the Advanced Automation System. The accuracy of a simple modification of the NAS algorithm that does meet this requirement is evaluated by means of a technique that avoids determination of the geometric relationship of the plane and the control jurisdiction.

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EXECUTIVE SUMMARY

There are two basic elements involved in the design of a stereographic projection system for an area control facility (ACF). One of these is the geometric relationship between an ellipsoidal model of the sea level surface of the earth and a plane surface on which stereographic images of points on the model surface can be displayed. The other is a projection algorithm that converts the altitude and radar measurements of the slant range and azimuth of an aircraft into a point on the plane that is an estimate of the actual stereographic representation of aircraft latitude and longitude. The display surface can be viewed as a plane tangent to a sphere with a center that is collocated with the center of the earth. The geometric relationship between the plane and the ellipsoidal earth model is specified by the radius of the sphere and the point of tangency. These two parameters determine the dilation characteristic on the control jurisdiction of the ACF. The structure of the projection algorithm pretty much determines the size of the projection error (i.e., the distance between the estimate of aircraft latitude and longitude and the actual stereographic representation of these angles). However, for a given projection algorithm, the error size does change with the geometric relationship between the earth model and the stereographic plane.

This report is concerned with the structure and performance of a projection algorithm developed at the Federal Aviation Administration Technical Center. Methods for achieving an optimal dilation characteristic are covered elsewhere. While the performance evaluation of the algorithm is based on the assumption that the geometric relationship between the earth model and the stereographic plane optimizes the dilation characteristic on the control jurisdiction, the evaluation methodology avoids the need to deal directly with this geometry. As a result, it provides an efficient means for testing algorithm performance for all possible radar site locations within the border of any ACF that might be commissioned in the Advanced Automation System.

1. INTRODUCTION.

Since the inception of the Design Competition Phase of the Advanced Automation System (AAS) program, a great deal of effort has been expended in the development of projection algorithms. In August 1985 the Government provided both the International Business Machines Corporation and the Hughes Aircraft Corporation with the details of a projection scheme developed at the Federal Aviation Administration (FAA) Technical Center. By the fall of 1986, both corporations were able to come up with algorithms of their own. This report discloses the results of computer experiments performed at the FAA Technical Center for the purpose of evaluating the FAA algorithm.

The objective of the FAA algorithm (reference 1) is the same as that of the projection algorithm currently employed in the National Airspace System (NAS) (i.e., to provide a stereographic representation of the latitude and longitude of an aircraft from reported altitude and radar measurements of slant range and azimuth). The FAA algorithm is a simple nonrepetitive procedure that involves the application of correction factors to the output of the NAS algorithm which by itself is unable to meet the accuracy requirements of the AAS. The algorithm is based on the fact that when the actual stereographic representation of the aircraft and the representation provided by the NAS algorithm are viewed as complex numbers then the differences between the magnitudes and the phases of the two numbers are approximately sinusoidal functions of azimuth (references 2 and 3) and the amplitudes of the sinusoids can be estimated by known functions of slant range and altitude. In essence, the FAA algorithm involves the addition of sinusoidal correction factors to the phase and magnitude of the output of the NAS algorithm.

The remainder of the report consists of seven sections. Section 2 is a brief description of some pertinent details regarding the structure of a stereographic projection system for an air traffic control facility supported by numerous surveillance radars. In section 3, a method is disclosed for evaluating the accuracy of a projection algorithm that is essentially independent of the location, size, and structure of the coverage region supervised by the facility. Section 4 is a description of the data used in the evaluation of the FAA projection algorithm. Numerical results are presented in section 5. Section 7 deals with the mechanics of the FAA algorithm which, as already indicated, is closely related to the projection algorithm currently employed in NAS. Details of the NAS algorithm appear in section 6. Concluding remarks can be found in section 8.

2. BACKGROUND.

Separation standards in NAS are maintained through control from the ground of the separation of aircraft in at least one of three dimensions, namely, altitude, latitude, and longitude. Aircraft

altitude is conveyed to the control facility by voice communication or transponder replies to radar interrogation messages. The remaining dimensions are obtained indirectly through some data processing scheme (i.e., the projection algorithm) that manipulates raw surveillance information (i.e., reported altitude and radar measurements of slant range and azimuth).

From an idealistic viewpoint the projection algorithm is supposed to provide a point on a plane surface (i.e., the system plane of the control facility) that is the stereographic representation of aircraft latitude and longitude with respect to some ellipsoidal earth model (e.g., the reference ellipsoid chosen for the North American Datum of 1983 (reference 4)) that approximates the geoid. The plane is tangent to a conformal sphere (i.e., a sphere with a center that is collocated with that of the ellipsoidal earth model). The point of tangency and radius of the sphere are chosen (or at least should be chosen) so as to optimize the dilation (reference 3, part II and reference 5) over the floor of the control jurisdiction (i.e., the portion of the surface of the earth model directly beneath the airspace that is under the supervision of the control facility). The control facility attempts to separate aircraft in one or both of the dimensions of latitude and longitude by keeping whatever points are provided by the projection algorithm a prescribed distance apart. However, data processing schemes with the same objective do not necessarily perform equally well and, regardless of the accuracy of the raw surveillence information, there is always some difference (i.e., a projection error) between the actual stereographic representation of an aircraft and the point representation provided by any practical projection algorithm. If the difference is large then there is some question as to what is being controlled (e.g., latitude and longitude or something else). If it is small then there is no such question.

The projection algorithm currently employed in NAS generates errors as large as 0.2 nautical mile (nmi) (references 2 and 3). In the case of the AAS the requirement is that the projection error shall not exceed 0.005 nmi (reference 6) for any aircraft under radar control within the jurisdiction of an area control facility (ACF). It is conceivable that a single ACF control jurisdiction could include the airspace under 60,000 feet and over a portion of the surface of the earth as large as the contiguous United States. Practically speaking, jurisdictions of this magnitude will only be encountered in ACFs charged with the control of oceanic air traffic, and the contiguous United States will, as now, be partitioned into much smaller regions. Nevertheless, the projection errors associated with any projection algorithm in compliance with the AAS specification must not exceed 0.005 nmi over extremely large volumes of airspace. While the specification is somewhat vague as to what is meant by a large volume, it seems that a sufficient volume for any practical purpose can be defined to be the airspace under 60,000 feet and over that portion of the surface of the earth

model that extends approximately 1815 nmi in any direction from a predetermined latitude and longitude (reference 5). In what follows data are presented that provide strong support for the contention that the FAA projection algorithm satisfies the 0.005 nmi error criterion over just such volumes of airspace.

As will be seen, it is possible to show that the FAA projection algorithm meets the 0.005 nmi error requirement of the AAS specification without resorting to a determination of the radius of the spherical support of the system plane or the location of the system plane point of tangency. On the other hand, both of these parameters must be evaluated in any operational environment. While this report does not address that problem, methods are available (reference 5) for determining an optimal radius for the conformal sphere that supports the system plane and an optimal location for the system plane point of tangency.

3. TEST METHODOLOGY.

Projection error is a function of many variables aside from those that describe the location of the aircraft relative to the radar from which slant range and azimuth are measured. Some of these define the location in absolute terms of the radar (e.g., radar site latitude and altitude). Others describe the location, size, and structure of the ACF as a whole (e.g., the radius of the conformal sphere that supports the system plane, the latitude of the point of tangency, and the distance measured in the system plane between the point of tangency and the stereographic representation of the radar site). While one can conduct a brute force evaluation of a projection algorithm by examining projection errors in the ACF system plane, it is clear that any such evaluation will be dependent upon factors unique to the location and overall structure of whatever ACF is involved in the testing procedure.

A better approach to projection algorithm evaluation is available that takes advantage of an upper bound on error that is derived from the simple bilinear transformation (i.e., the transformation equation (references 2 and 3)) relating two distinct stereographic representations of the same latitude-longitude pair. These are the representation in the system plane and the representation in the plane tangent to a conformal sphere at the latitude and longitude of the radar site. The latter plane is sometimes referred to as the local radar plane, and the sphere that supports the local plane is rarely the same as the sphere that supports the system plane. In fact, the radius of the conformal sphere supporting the local plane is (or should be) equal to the distance between the center of the ellipsoidal earth model and the point on the surface of the model at the latitude and longitude of the radar site. As shown in appendix A, there exists an upper bound on the error induced in the system plane by a projection error e_L in the local plane that can be expressed as a function of the local plane error and the dilation design

constant n (i.e., the design goal for the median value of the dilation on the floor of the control jurisdiction (reference 5)). Indeed, if e_L is less than 1 nmi and the distance in the local plane separating the tangency point from the stereographic representation of the illuminated radar target (i.e., the target ground range) is at most 205 nmi then the distance e_B derived from the relationship

$$e_B = 1.094\ ne_L \tag{1}$$

is an upper bound on the corresponding error induced in the system plane.

The bound defined by equation (1) is valid for any ACF that might be commissioned in the AAS. It is also independent of those variables that characterize the location and overall structure of any such ACF. Thus, the bulk of the work involved in algorithm evaluation can be reduced to an examination of errors in the local plane. In effect, it is only necessary to determine the maximum error that the algorithm can induce in the local plane and then compute the corresponding error bound. If the result is less than 0.005 nmi when the design constant is set equal to 1 then it is certain that the algorithm meets the AAS accuracy requirement. Needless to say, if the maximum error in the local plane is 1 nmi or more then the error bound equation is invalid and one must either derive another bound or else reject the algorithm on the basis of the local plane data. While the derivation of another error bound relationship does not in itself present a serious problem, outright rejection of the algorithm is the appropriate action to take in the face of excessive local plane errors.

4. EXPERIMENTAL SETUP.

Determination of the maximum projection error in the local plane is not a mathematically tractable problem. It is only possible to obtain an estimate of the local plane maximum from empirical data. Consequently, it is necessary to obtain local plane error samples corresponding to enough aircraft positions relative to the radar and enough radar locations to establish beyond a reasonable doubt that the sample maximum is an accurate estimate of the actual local plane maximum.

The radar locations used in the generation of the error data reported here correspond to the 21 altitude-latitude pairs that can be formed from 3 altitudes and 7 latitudes relative to the reference ellipsoid chosen for the North American Datum of 1983. The altitudes are 0, 5,000, and 10,000 feet above the surface of this earth model (i.e., an ellipsoid with an equatorial radius of 3443.919 nmi and a polar radius of 3432.372 nmi). The latitudes extend from 15° north geodetic latitude to 75° north geodetic latitude in 10° increments.

Aircraft location relative to the radar can be defined in terms of altitude, azimuth, and slant range. Another (but equally good) specification of aircraft location is used here. Specifically, relative location is defined in terms of altitude and the stereographic representation in the local plane of aircraft latitude and longitude. The local plane representation of latitude and longitude is commonly expressed as a complex number with respect to a Cartesian coordinate system in which the local plane point of tangency is the origin and the imaginary axis is the stereographic representation of the meridian plane passing through the radar site (references 2 and 3). In the case of aircraft locations in the meridian plane the phase angle of the complex number is identical to the complement of azimuth. For other aircraft locations it is close to, but not necessarily the same as, the complement of aircraft azimuth angle. As already indicated, the magnitude of the complex number conforms to the usual notion of aircraft ground range.

From considerations of symmetry about the meridian plane containing the radar, it is to be expected that if the dimension of altitude is unaltered then the magnitude of the projection error in the local plane associated with any complex number representation of aircraft latitude and longitude will be identical to the magnitude of the error associated with the negative of its complex conjugate. The NAS and FAA projection algorithms fulfill this expectation. Consequently, it is redundant to consider complex number representations of aircraft latitude-longitude pairs in both the right and left half planes. For this reason the data reported here is limited to aircraft locations for which the phase angles extend from -90° to +90° (i.e. azimuth angles are less than or equal to 180°).

Two groups of relative aircraft locations, expressed in terms of ground range-phase angle-altitude triplets, were used in the generation of the projection error data. One group (i.e., the short range group) is characterized by the 20 ground ranges that extend from 0.25 nmi to 5.00 nmi in 0.25 nmi increments. The other (i.e., the long range group) is characterized by the 21 ground ranges extending from 5 nmi to 205 nmi in increments of 10 nmi. In each group, the phase angle-altitude pairs coupled with each ground range are the same. Specifically, 481 phase angle-altitude pairs were constructed from the 37 phase angles extending from -90° to +90° in 5° increments and the 13 aircraft altitudes extending from 0 to 60,000 feet in 5,000-foot increments. Consequently, there are 9,620 relative aircraft locations in the short range group and 10,101 locations in the long range group.

Projection errors were generated by means of the following six-step procedure using a software package constructed by the MITRE Corporation (reference 7).

Procedure 1.

a. A radar location is selected in terms of an altitude-latitude pair. Since longitude is missing, this does not provide a complete specification of the position of the radar. On the other hand, projection error is independent of the longitude of the radar site. Consequently, without loss of generality, the altitude-latitude pair together with a longitude of 0° can be used to specify the location of the radar.

b. A triplet is selected from one or the other of the two groups of relative aircraft locations.

c. The latitude and longitude of the radar site together with the aircraft ground range and phase angle are used to compute the geodetic latitude and longitude of the aircraft. This result and aircraft altitude completely describe the location of the aircraft relative to the ellipsoidal earth model.

d. Slant range and azimuth are computed from the positions of the radar and the aircraft with respect to the earth model.

e. The slant range, azimuth, and altitude of the aircraft are input to the projection algorithm, and the algorithm in turn outputs an estimate of the stereographic representation in the local plane of aircraft latitude and longitude.

f. Projection error is computed as the euclidean distance between the complex formulation of the estimate and the complex number associated with the ground range and phase angle of the aircraft.

An error generated in this fashion is here classified as a long range error or a short range error depending upon the source (i.e., short or long range group) of the ground range-phase angle-altitude triplet.

The reported errors are those that are associated with aircraft locations within the radar coverage region. While the total number of short range errors generated from all 21 radars is 202,020 and the corresponding number of long range errors is 212,121, some of these are associated with aircraft locations that are outside of covered airspace (i.e., in the cone of silence or else below the radar horizon). In this report, the cone of silence is taken to be that portion of airspace for which the radar elevation angle exceeds 85°. In addition, refraction effects are ignored and so the airspace in the shadow of the earth below the line-of-sight horizon is also considered to be outside the coverage region. As a matter of coincidence rather than contrivance, it turns out that the number of short range errors associated with aircraft locations inside covered airspace is the same as the number of long range errors generated from covered aircraft locations (i.e., 186,192).

5. RESULTS.

Table 1 displays the maximum of all projection errors (i.e., both long and short range errors) in the local plane that were generated from aircraft locations in covered airspace. Also shown is the corresponding upper bound

TABLE 1. MAXIMUM ERROR ESTIMATE

Algorithm	Estimate of Maximum Error in Local Plane (nmi)	Error Bound (nmi)
FAA	0.00310	0.00339

(derived from equation (1)) on error in the ACF system plane. Needless to say, the bound is less than the 0.005 nmi error limit of the AAS specification.

Figures 1 and 2 illustrate the maximum of all projection errors observed in the local plane that were generated from aircraft locations with the same ground range. As indicated, the maximum of all observed errors, regardless of range, was generated at the maximum ground range (i.e., 205 nmi). Also, figures 1 and 2 demonstrate that the accuracy of the FAA algorithm improves as aircraft ground range decreases. Suprisingly enough, the same cannot be said for every projection algorithm.

The fraction of all observed errors in the long range data that exceed a prescribed error bound is displayed as a function of the bound in figures 3 and 4. For example, only 10 percent of the long range errors associated with the FAA algorithm exceed 0.0008 nmi. As shown in figure 1 all of the short range errors are less than 0.0001 nmi and it turns out that only 0.01613 percent of these exceed 0.00005 nmi.

6. NAS ALGORITHM.

A spherical model of the mean sea level surface of the earth was used in the development of the NAS projection algorithm. In the case of a spherical earth model the complement of aircraft azimuth angle A is the same as the phase angle of the complex number that is the stereographic representation in the local plane of aircraft latitude and longitude. Moreover, the magnitude r of this number can be expressed as a simple function of aircraft slant range S, aircraft altitude H above mean sea level, radar altitude H_R above mean sea level, and the radius E of the earth model. Specifically,

$$r = Rx^{-1} \tag{2}$$

where

$$R = [S^2 - (H - H_R)^2]^{1/2} \tag{3}$$

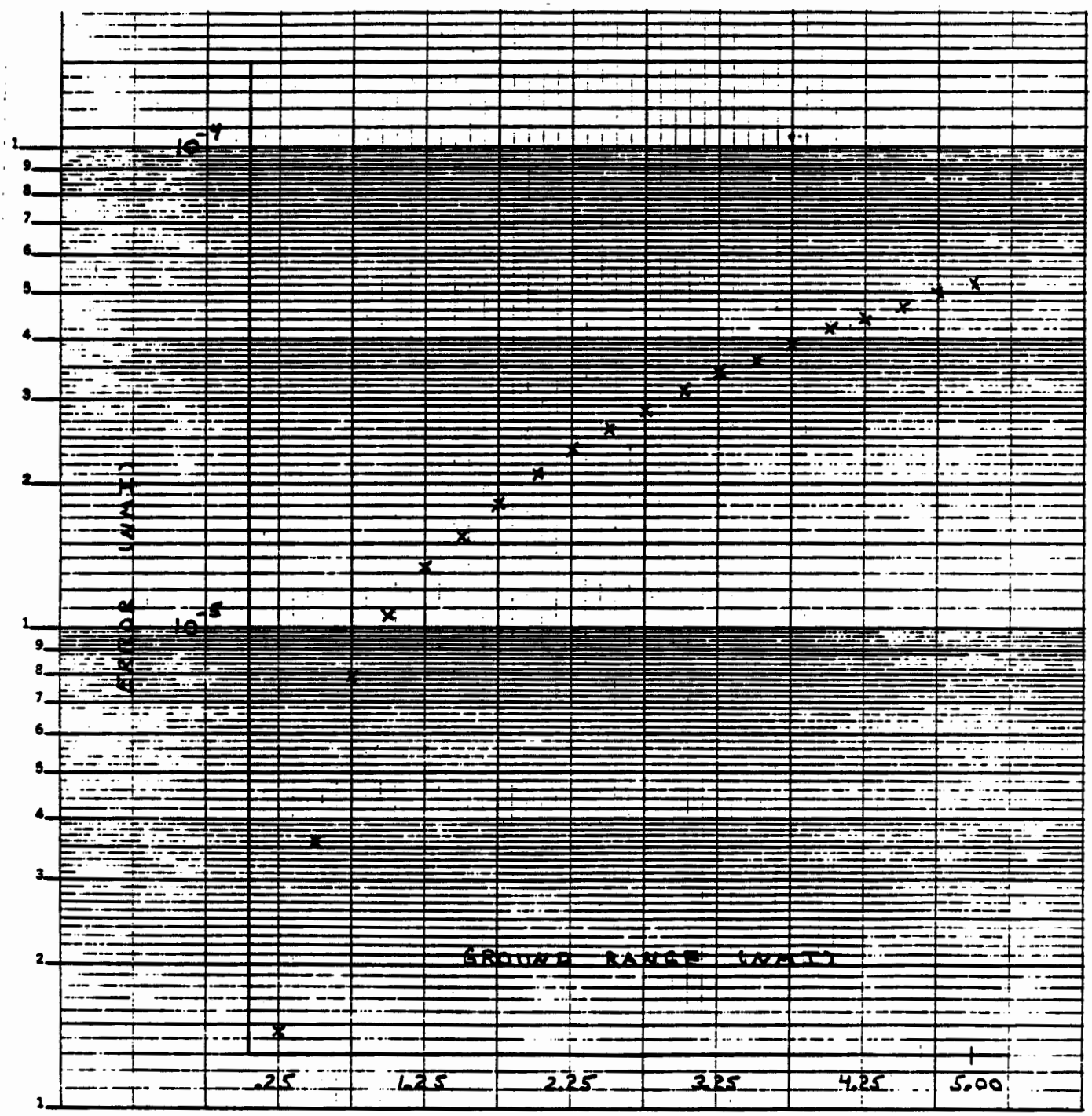

FIGURE 1. MAXIMUM POSITION ERROR -- SHORT RANGE DATA

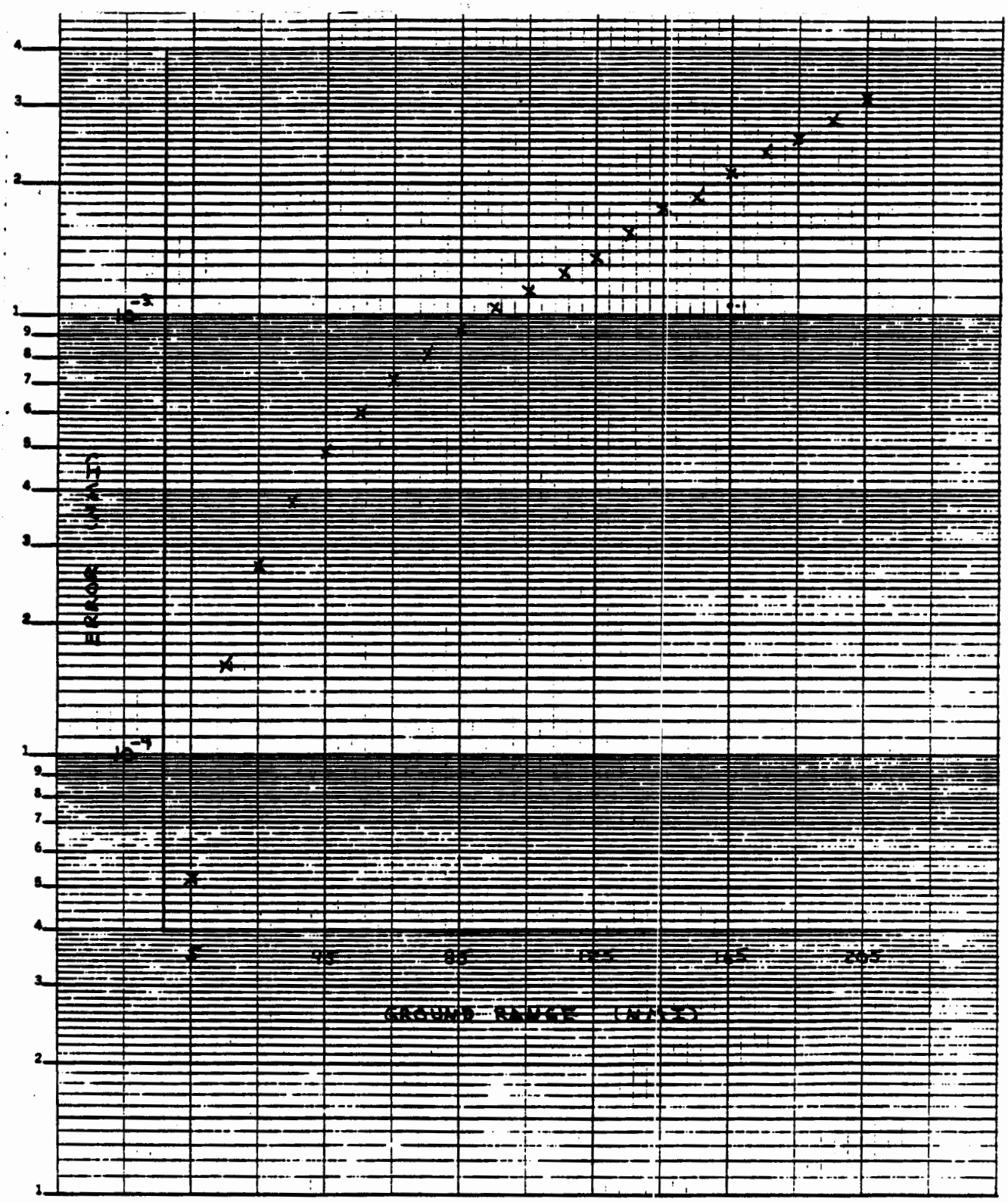

FIGURE 2. MAXIMUM POSITION ERROR -- LONG RANGE DATA

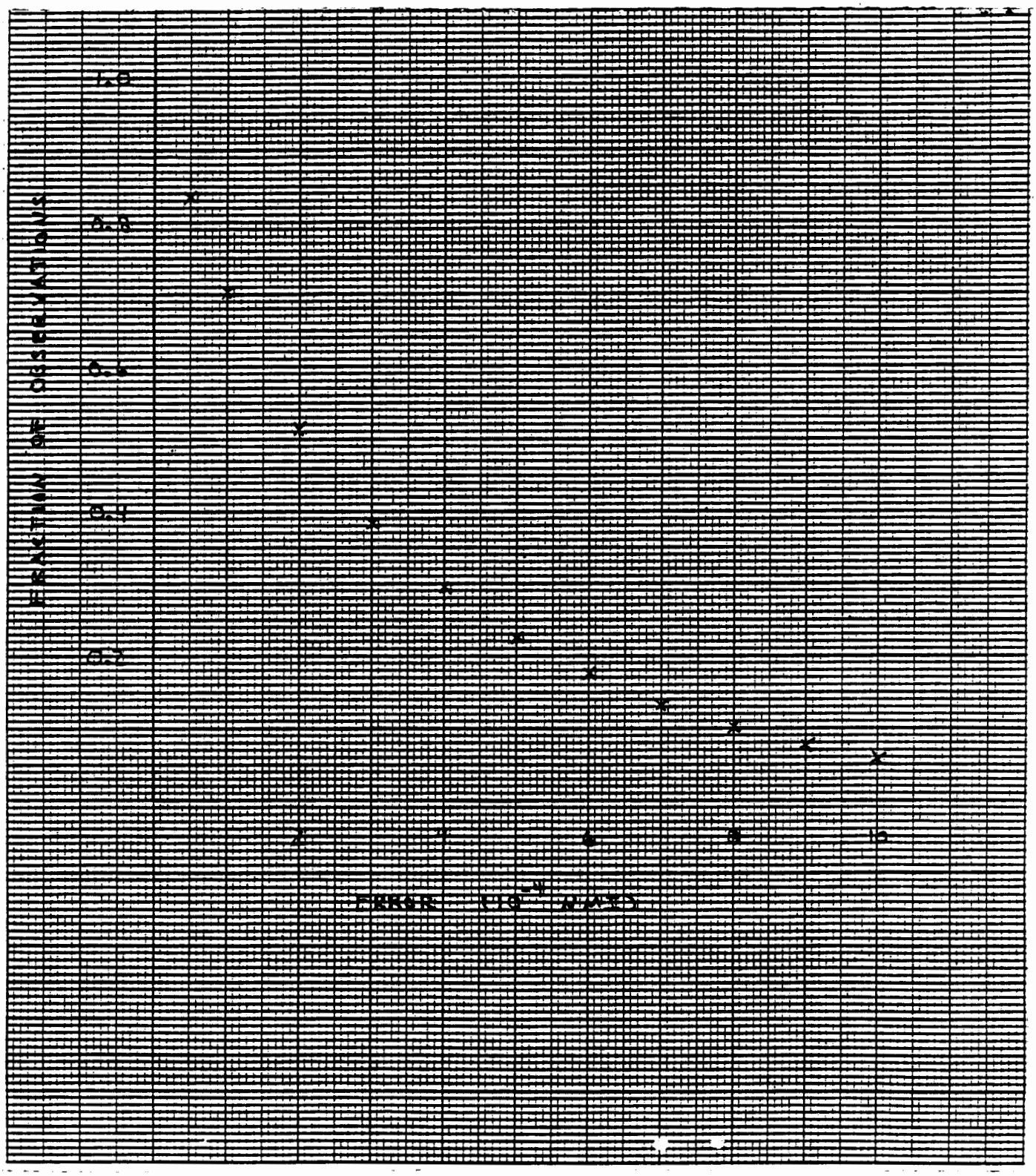

FIGURE 3. ERROR DISTRIBUTION -- LONG RANGE DATA

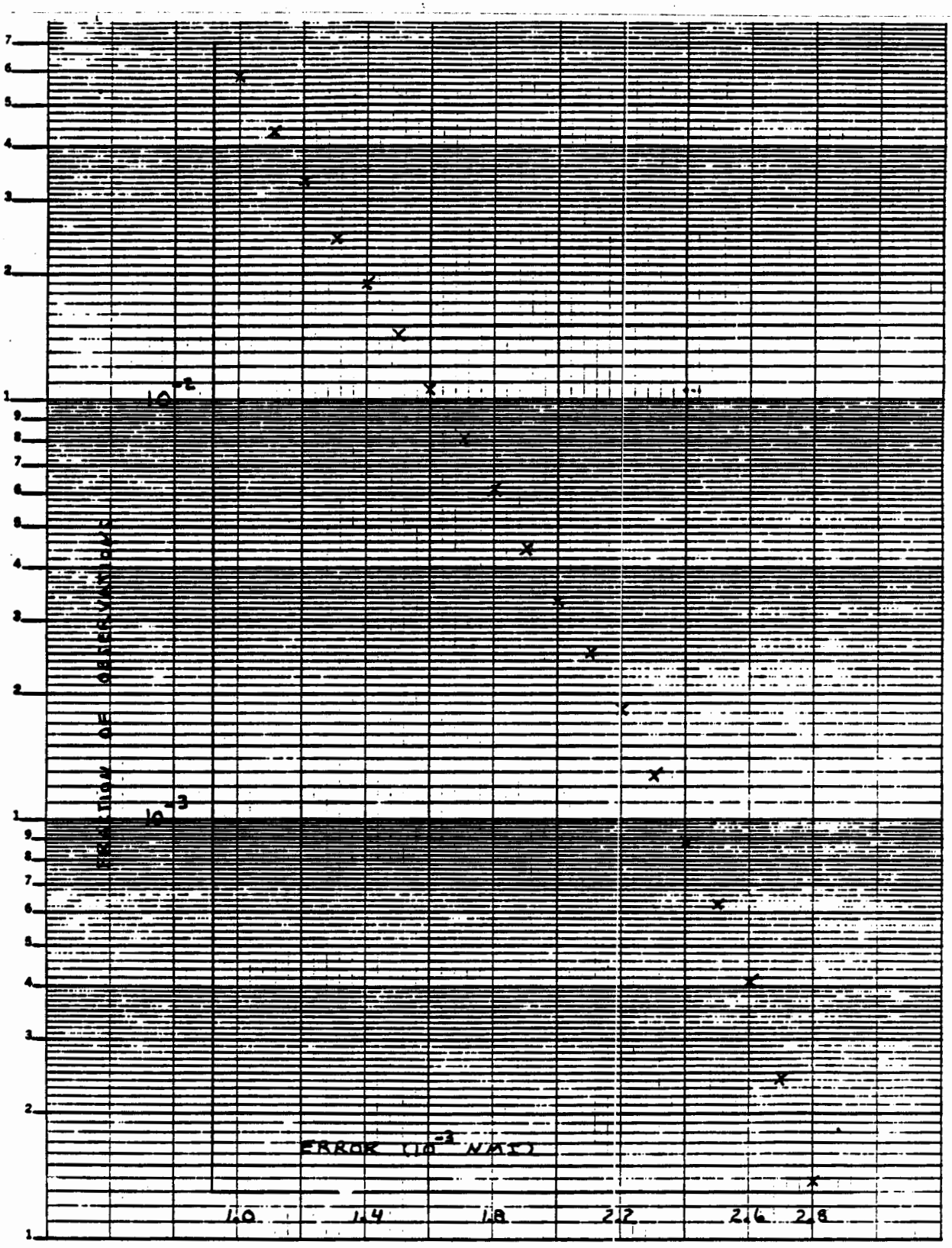

FIGURE 4. ERROR DISTRIBUTION -- LONG RANGE DATA (CONTINUATION)

and

$$x = [2E]^{-1} [(2E + H + H_R)^2 - S^2]^{1/2} . \quad (4)$$

Thus, in the case of a spherical earth model the complex number

$$Z_N = r \exp[j(pi/2 - A)] \quad (5)$$

is the stereographic representation in the local plane of aircraft latitude and longitude. In essence, this is the output of the NAS algorithm.

If the NAS algorithm is to be employed in conjunction with an ellipsoidal earth model then, since the radius of the model changes with latitude, some refinement of the definitions of the parameters H, H_R, and E is in order. The altitude of an object becomes the distance separating the object from the point on the surface of the model at the geodetic latitude and longitude of the object and E becomes the distance from the center of the earth to the point on the model surface at the geodetic latitude and longitude of the radar. Unfortunately, under the assumption of an ellipsoidal earth model there are numerous aircraft locations relative to the radar for which Z_N differs substantially from the complex number z that is the stereographic representation in the local plane of aircraft latitude and longitude. For this reason, the NAS algorithm cannot serve as the AAS projection algorithm.

7. FAA ALGORITHM.

The FAA algorithm can be viewed as a procedure by which correction factors dr and dp are added to the magnitude and phase, respectively, of the NAS estimator to construct the complex number

$$Z_F = (r + dr)\exp[j(pi/2 - A + dp)] \quad (6)$$

(i.e., the FAA estimator). The correction factors are defined by the formulas

$$dr = a_F r^2 \cos(A) \quad (7)$$

and

$$dp = b_F r \sin(A) \quad (8)$$

where a_F and b_F are dependent on the altitude and latitude of the radar site (references 2 and 3).

Values for the parameter a_F can be obtained from the application of the least squares method to the sum of squares of terms representing errors in the magnitude of the FAA estimator. The remaining parameter can be evaluated by the same method using errors in the phase of the estimator. The computation for a_F is

particularly simple in the case where each term of the sum corresponds to an aircraft latitude-longitude pair represented by a complex number on the positive half of the imaginary axis. Specifically, if z is such a number then the corresponding aircraft azimuth angle is Ø rad. Consequently, the difference between the magnitude of z and the magnitude of the FAA estimator is $r + a_F r^2 - mag(z)$. The computation for b_F can be simplified by using complex numbers on the positive half of the real axis. Indeed, if z is a real and positive number that is no more than a few hundred nmi then the corresponding aircraft azimuth angle will be very close (but not equal) to pi/2 rad and the difference between the phases of the estimator and z is just the phase of the former. In other words, the difference is essentially the same as $pi/2 - A + b_F r$. The details of the least squares solutions for a_F and b_F derived from these formulations of magnitude and phase errors are provided in the following procedure.

Procedure 2. Assignment of Values to a_F and b_F.

It is assumed that the location of the radar has been specified in terms of an altitude, a geodetic latitude, and a longitude of Ø°. It is also assumed that a stack of aircraft altitude-real positive number pairs of the form (H,P) has been created where P represents the magnitude of the complex representation on the imaginary or real axis of an aircraft latitude-longitude pair.

a. Initialize sums S_{1L}, S_{1R}, S_{2L}, and S_{2R} to Ø.

b. Using H and P in the pair at the top of the stack and the radar location parameters, compute the NAS ground range r_1 associated with the aircraft altitude-complex number pair (H,jP).

c. Increase S_{1L} by $(r_1)^4$ and S_{1R} by $(r_1)^2(P - r_1)$.

d. Using the radar location parameters, compute the azimuth angle A and NAS ground range r_2 associated with the aircraft altitude-complex number pair (H,P).

e. Increase S_{2L} by $(r_2)^2$ and S_{2R} by $r_2(A - pi/2)$.

f. Decrease the size of the stack by 1 by removing the pair at the top of the stack.

g. If the stack is not empty then go to step b. Otherwise, assign the value S_{1R}/S_{1L} to the parameter a_F and the value S_{2R}/S_{2L} to the parameter b_F.

Values assigned to the parameters a_F and b_F for the purpose of generating the reported projection errors were obtained by means of procedure 2. The stack was created from (H,P) pairs constructed from a single altitude (i.e., 3Ø,ØØØ feet) and the 128 ground ranges, other than Ø, that are integer multiples of

205/27 nmi not exceeding 205 nmi. The values assigned to the parameters are displayed in tables 2 and 3. The variation of parameter values with altitude is small compared to the variation with latitude.

TABLE 2. FAA ALGORITHM PARAMETER a_F (10^{-7} nmi^{-1})

Radar Latitude (deg)	Radar Altitude (ft) 0	5,000	10,000
15	2.378802	2.335961	2.293141
25	3.677897	3.646522	3.615161
35	4.511287	4.495151	4.479022
45	4.765510	4.766553	4.767998
55	4.403381	4.421472	4.439553
65	3.471638	3.504587	3.537520
75	2.093987	2.137808	2.181608

TABLE 3. FAA ALGORITHM PARAMETER b_F (10^{-7} nmi^{-1})

Radar Latitude (deg)	Radar Altitude (ft) 0	5,000	10,000
15	-2.427211	-2.427211	-2.427211
25	-3.721020	-3.721021	-3.721021
35	-4.568345	-4.568346	-4.568348
45	-4.866189	-4.866191	-4.866193
55	-4.577112	-4.577115	-4.577117
65	-3.734452	-3.734455	-3.734457
75	-2.439023	-2.439025	-2.439027

While the altitude effect on the phase of the FAA estimator is clearly insignificant, changes in parameter values for a_F with altitude can materially affect the projection error. For example, suppose that two estimators are constructed for the same radar site by assigning distinct values v_1 and v_2 to the parameter a_F. The euclidean distance separating the estimators is $r = mag(v_1 - v_2)$. Suppose now that the radar site is at sea level, the site latitude is 75°, and the values v_1 and v_2 are those prescribed by table 2 for that latitude at sea level and 10,000 feet, respectively. In the case of an aircraft ground range of 205 nmi the corresponding distance between estimators is in the neighborhood of 0.000368 nmi. Referring to figure 2, this is greater than 10 percent of the maximum projection error associated with the FAA algorithm at a ground range of 205 nmi. On the other hand, the altitude effect is not so serious that it will lead to projection errors in excess of the AAS 0.005 nmi error limit.

The local plane estimate of the stereographic representation of aircraft latitude and longitude must be mapped into the ACF system plane. From the point of view of computational speed this

mapping should be implemented in terms of arithmetic operations on the real and imaginary parts of the local plane estimate. The real and imaginary parts of the FAA estimator are

$$re(Z_F) = (r + dr)[\sin(A)\cos(dp) - \sin(dp)\cos(A)] \qquad (9)$$

and

$$im(Z_F) = (r + dr)[\cos(A)\cos(dp) + \sin(A)\sin(dp)]. \qquad (10)$$

Unfortunately, these expressions are rather complicated. This complication can be removed by exploiting the fact that the correction factors dr and dp are very small numbers. In particular, little is lost in the formulas for the real and imaginary parts of Z_F by replacing cos(dp) with 1 and sin(dp) with dp and then throwing away any terms in the resulting expressions that contain the product of dr and dp as a factor. The final result can be viewed as a complex number W_F for which

$$re(W_F) = r[\sin(A) - r([b_F - a_F]/2)\sin(2A)] \qquad (11)$$

and

$$im(W_F) = r[\cos(A) - r([b_F - a_F]/2)\cos(2A)] + [(a_F + b_F)/2]r^2. \qquad (12)$$

In fact, the projection error data reported here is representative of the euclidean distance between W_F and z rather than that between Z_F and z. However, it has been verified in other computer experiments that the reported data also constitute, as expected, an accurate description of the accuracy of the estimator Z_F. Further simplification of the estimator is possible through the elimination of the factor $[(a_F + b_F)/2]r^2$ in (12). As shown in appendix B, the maximum projection error in the system plane associated with the estimator resulting from this simplificatiion is less than the 0.005 nmi requirement of the AAS specification. Needless to say, the simplification effectively reduces the two parameters a_F and b_F to one, i.e., half of the difference $b_F - a_F$.

8. CONCLUSIONS.

The reported data support the contention that the FAA algorithm satisfies the AAS requirement that the projection error not exceed 0.005 nmi for any aircraft in an ACF coverage region. This conclusion is based on the assumption that the spherical support radius and tangency point for the ACF system plane are optimal in the sense that the values assigned to these parameters minimize the variation of the dilation on the floor of the control jurisdiction about a predetermined constant. In addition, the mathematical structure of the algorithm is simple. As a result, efficient computer implementations of the algorithm

can be expected to yield high speed computation of system plane representations of aircraft latitude and longitude with mimimal allocation of storage space for adaptation parameters.

While the reported data support the contention that the FAA algorithm satisfies the AAS accuracy requirement, it might be argued that the sample of aircraft locations relative to the radar employed in the study is flawed. For example, the data indicate that the maximum projection error at a given ground range increases monotonically with the range. On the other hand, the density of locations decreases with distance from the antenna. In fact, the azimuthal distance between adjacent locations at the same sample ground range is 5°. This is equivalent to an arc length of 17.9 nmi between aircraft locations at the maximum ground range of 205 nmi. Accordingly, future investigations of the accuracy of the algorithm might do well to concentrate on the error in the neighborhood of the maximum ground range and use a smaller azimuthal separation between adjacent sample aircraft locations at the same range.

It is worthwhile noting that the maximum error generated by the algorithm is dependent upon the values assigned to the parameters a_F and b_F. If procedure 2 is used to make the assignment then the maximum error in the local plane is the value reported in table 1. If some other assignment procedure is used then the maximum local plane error may be greater or less than 0.0031 nmi. For example, procedure 2 is just the result of applying the method of least squares to points on the real and imaginary axes of the local plane coordinate system. The parameters can also be evaluated by means of an application of the least squares method to points on and off the axes. Another approach to the assignment problem is to recognize that the correction factors (7) and (8) are monotonic functions of r that vanish when r is zero. This means that r and pi/2 - A are pretty good estimates of the amplitude and phase of the stereographic image of a target close in to the antenna. Consequently, it might be possible to reduce the maximum error by means of a least squares fit of the amplitude and phase of the complex variable (6) that is restricted to points in the local plane at large ground ranges (e.g., 150 nmi - 205 nmi). The hope is that the restriction will lower the error when the range is long and the monotonic behavior of the correction factors will automatically take care of the situation at shorter ranges.

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APPENDIX A. BOUND ON THE SYSTEM PLANE ERROR

There exists an upper bound on the error induced in the system plane by a projection error e_L in the local plane that can be expressed in terms of the radius E of the conformal sphere that supports the local plane, the radius E_r of the sphere that supports the system plane, the euclidean distance w_o measured in the system plane between the tangency point and the stereographic representation of the geodetic latitude and longitude of the radar site, and the euclidean distance z separating the local plane tangency point from the stereographic representation in the local plane of the geodetic latitude and longitude of the target. The bound is the ratio of

$$N = (E_r/E)\{1 + [w_o/(2E_r)]^2\}e_L \qquad (A-1)$$

to

$$D = [1 - u(z,e_L)]^2 \qquad (A-2)$$

where

$$u(z,e_L) = [w_o/(2E_r)][(z + e_L)/(2E)]. \qquad (A-3)$$

It can be easily derived in a straight forward manner from known results (Mulholland, R. and Stout, D., Numerical Studies of Conversion and Transformation in a Surveillance System Employing a Multitude of Radars - Part II, Astia Document AD A072-086, National Technical Information Service, Springfield, VA, April 1979, equation (58) and the appendix). The bound can be re-expressed in terms of the angle V subtended at the center of the earth model by the point of tangency of the system plane and the conformal representation of the radar (i.e., the point on the surface of the spherical support of the system plane defined by the conformal latitude and longitude of the radar). This is a direct result of the fact that the ratio of w_o to $2E_r$ is just the tangent of V/2.

The number N cannot exceed

$$N_o = n(a/b)[1 + \tan^2(G_c/2)]e_L \qquad (A-4)$$

where a and b are the equatorial and polar radii of the reference ellipsoid chosen for the North American Datum of 1983, n is the dilation design constant, and G_c is 30.250282°. In addition,

$$u_o(z,e_L) = [(z + e_L)]/(2b)\tan(G_c/2) \qquad (A-5)$$

is an upper bound on the parameter u. It is known that the optimal choice for the radius of the spherical support of the system plane for any ACF that might be commissioned in the AAS cannot exceed the product of a and n (Mulholland R., Optimization of the Dilation Characteristic on the Control Jurisdiction of an Area Control Facility, Report No. DOT/FAA/CT-TN87/39, FAA

Technical Center, Atlantic City Airport, NJ, August 1987). Also, it is known that the maximum angle subtended at the center of the earth by the system plane point of tangency and the conformal representation of any point in the floor of a control jurisdiction meeting the AAS design limit on the size of an ACF coverage region should not exceed the sum of 30.25° and 0.000282°. Thus, the tangent of $G_c/2$ is an upper bound for the ratio $w_c/(2E_r)$. Moreover, the product of a and n is an upper bound on the parameter E_r. Finally, since E is just the distance separating the center of the earth from the point on the model surface at the geodetic latitude and longitude of the radar site, it is clear that E is not less than b.

The number D is not less than

$$D_c = [1 - u_c(205,1)]^2, \qquad \text{(A-6)}$$

and so the ratio of N to D cannot exceed

$$e_B = N_c/D_c. \qquad \text{(A-7)}$$

If the ground range z is at most 205 nmi and the projection error e_L is less than 1 nmi then it is clear that $u_c(z,e_L)$ is at most $u_c(205,1)$. Since $u_c(205,1)$ is less than 1 it follows that D_c bounds D from below.

APPENDIX B. BOUND ON THE PROJECTION ERROR ASSOCIATED WITH A MODIFICATION OF THE FAA ALGORITHM

If U is the complex number defined by the equations

$$re(U) = re(W_F) \tag{B-1}$$

and

$$im(U) = im(W_F) - [(a_F + b_F)/2]r^2 \tag{B-2}$$

then U and the FAA estimator W_F satisfy the inequality

$$mag(U - W_F) \leq (0.20 \times 10^{-7})r^2 \tag{B-3}$$

under the experimental setup of section 4. Indeed, equations (11) and (12) imply that $mag(U - W_F)$ is the same as the product of r^2 and $(a_F + b_F)/2$. The inequality (B-3) follows directly from the implication of tables 2 and 3 to the effect that the magnitude of the sum of a_F and b_F is less than 0.40×10^{-7} nmi^{-1}.

If r is expressed in nmi then the inequality (B-3) can be used to show that the local plane error in nmi associated with the estimator U cannot exceed the bound

$$B(r) = 0.00310 + (0.20 \times 10^{-7})r^2 \tag{B-4}$$

under the experimental conditions described in section 4. Table 1 implies that the maximum error associated with the FAA estimator W_F is 0.00310 nmi. The bound B(r) follows directly.

The parameter r cannot be greater than 205.025 nmi under the experimental setup of section 4. Specifically, equations (6) and (7) imply that the magnitude of the FAA estimator is the sum of r and $a_F r^2 \cos(A)$. Since table 2 shows the parameter a_F to be a positive number no greater than 5×10^{-7} nmi^{-1} and table 1 shows the maximum local plane error associated with the FAA estimator to be 0.00310 nmi, it follows that

$$r - (5 \times 10^{-7})r^2 \leq 205.00310 \tag{B-5}$$

under the experimental conditions of section 4. Otherwise, the maximum error for the FAA estimator would be larger than the observed maximum of 0.00310 nmi. The left side of the inequality (B-5) is an increasing function of the parameter r for values of the parameter that are less than 10^6 nmi, and it equals the right side of the inequality when the argument is 205.0241174 nmi. As a result, the parameter r must take on values less than 205.025 nmi under the cited experimental conditions.

The system plane error associated with the estimator U is at most 0.00431 nmi. Indeed, the number B(205.025) obtained from (B-4) when r is 205.025 nmi is less than 1. Thus, the bound (1) is

valid and the system plane error cannot exceed (1.094)B(205.025) nmi.

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